A RATIONAL APPROACH TO IRRATIONAL NUMBERS

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One of the great discoveries of the Pythagorean era was the fact that not all positive numbers are "commensurable", that is, expressible as a fraction a/b where a and b are natural numbers. In modern terminology the Pythagoreans discovered irrational numbers. The first and perhaps the simplest example of such a number was $\sqrt{2}$ and you probably know the ingenious argument by which Pythagoras proved its irrationality. Suppose that $\sqrt{2}$ was rational, $\sqrt{2} = a/b$ say, where a and b may be assumed to be relatively prime. Squaring the equation and multiplying through with b^2 we get $a^2 = 2b^2$ which can only hold if a is even, a = 2c say. But then $4c^2 = 2b^2$ or $b^2 = 2c^2$, that is b is also even, in contradiction to the assumption that a and b have no common divisor greater than 1. This same argument works for all numbers of the form $\sqrt{d}$ where d is any positive integer not a perfect square.

We shall now prove the irrationality of $\sqrt{2}$ by a very different, and you may say far more difficult method which, on the other hand, can be applied to a wider class of numbers than the method of Pythagoras; $\sqrt{2}$ will merely serve as a convenient illustration of the method. The principle upon which many proofs of irrationality rest is the following theorem.

Theorem. Let α be a real number. If there is an infinite sequence of fractions a(n)/b(n) in lowest terms, with denominators $b(1) < b(2) < \ldots < b(n) < \ldots$, such that $b(n)\alpha - a(n)$ tends to 0 as n increases, then α is irrational.

Notice that the theorem embodies a rather sophisticated analytical principle, requiring the existence of an infinity of fractions with a certain property whereas the proof of Pythagoras is perfectly "finite".

To prove the theorem, suppose $\alpha = a/b$ is rational and take a sequence of fractions a(n)/b(n) in lowest terms with denominators $b(1) < \ldots < b(n) < \ldots$. Since the denominators b(n) increase, we can choose m so large that b(m) > b. Now

$$\alpha - \frac{a(n)}{b(n)} = \frac{a}{b} - \frac{a(n)}{b(n)} = \frac{ab(n) - ba(n)}{bb(n)}.$$

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For $n \geqslant m$, this expression is non-zero because we have $b(n) > b$ and so $a/b \neq a(n)/b(n)$. Thus the numerator $ab(n) - ba(n)$ is a non-zero integer and, in particular, $|ab(n) - ba(n)| \geqslant 1$. Hence

$$|\alpha - a(n)/b(n)| \geqslant 1/bb(n),$$

or multiplying through by $b(n)$,

$$|b(n)\alpha - a(n)| \geqslant 1/b.$$

This holds for all $n \geqslant m$, and so $b(n)\alpha - a(n)$ cannot tend to 0. It can only tend to 0 if α is irrational, and the theorem follows.

Incidentally the converse of the theorem is also true, namely if α is irrational, then there always exists a sequence of fractions satisfying the conditions of the theorem. Can you prove it?

What have we gained by our theorem? It tells us that in order to demonstrate the irrationality of a number α, all we have to find is a sequence of fractions $a(n)/b(n)$ which satisfy the conditions of the theorem. We shall construct such a sequence for $\alpha = \sqrt{2}$.

Consider the sequence $u(n)$ defined by the recursion

$$u(n) = 2u(n-1) + u(n-2) \quad (n \geqslant 2) \tag{1}$$

Such a sequence is uniquely determined provided that we specify the "initial values" $u(0)$ and $u(1)$. Take for instance $u(0) = 0$ and $u(1) = 1$. Then $u(2) = 2$, $u(3) = 5$, $u(4) = 12$, and so on. We shall show that $u(n)/u(n-1)$ tends to $1 + \sqrt{2}$ and for this purpose, we shall solve the recursion explicitly. Although this is not the simplest way to calculate the limit of $u(n)/u(n-1)$, the explicit expression will give us further useful information.

There is a general method for finding a formula for $u(n)$ when it satisfies a recursion of the type (1). Let us try to satisfy (1) by an expression of the shape

$$u(n) = c\lambda^n \tag{2}$$

where c and λ are non-zero numbers. We can indeed satisfy (1) by such an expression provided that λ is chosen appropriately. To determine this value of λ, we substitute (2) into (1), giving $c\lambda^n = 2c\lambda^{n-1} + c\lambda^{n-2}$, and after dividing through by $c\lambda^{n-2}$, this is $\lambda^2 = 2\lambda + 1$. So we obtain two possible values for λ; let us call them

$$\lambda = 1 + \sqrt{2} \quad \text{and} \quad \mu = 1 - \sqrt{2}.$$

It is now easy to check that for any given values of a and b

$$u(n) = a\lambda^n + b\mu^n \tag{3}$$

is a solution of (1). (Prove it.) Moreover, this is the general solution, for given the initial values $u(0)$ and $u(1)$, we can always find a and b so that these initial values are given by (3), namely

$$u(0) = a + b, \quad u(1) = a\lambda + b\mu.$$

Solving for a and b, we obtain

$$a = (u(1) - u(0)\mu)/(\lambda - \mu), \quad b = (u(0)\lambda - u(1))/(\lambda - \mu).$$

In particular, if $u(0) = 0$ and $u(1) = 1$, we get $a = -b = 1/2\sqrt{2}$, and so the solution (3) becomes

$$u(n) = (\lambda^n - \mu^n)/2\sqrt{2} \quad (n = 0, 1, 2, \ldots).$$

This formula enables us to calculate the limit of u(n)/u(n − 1). Indeed,

$$\frac{u(n)}{u(n-1)} = \frac{\lambda^n - \mu^n}{\lambda^{n-1} - \mu^{n-1}} = \frac{\lambda - \mu(\mu/\lambda)^{n-1}}{1 - (\mu/\lambda)^{n-1}}.$$

Here, $(\mu/\lambda)^{n-1} = \{(1-\sqrt{2})/(1+\sqrt{2})\}^{n-1}$ tends to 0 as n increases. (Why?) Consequently, the terms involving $(\mu/\lambda)^{n-1}$ disappear in the limit as n tends to infinity and we have

$$u(n)/u(n-1) \rightarrow \lambda = 1 + \sqrt{2} \quad \text{as } n \rightarrow \infty,$$

as required.

Set now a(n) = u(n) and b(n) = u(n − 1) for n = 1, 2, 3, ... Then we have a sequence of fractions a(n)/b(n) which tends to $\lambda = 1 + \sqrt{2}$. Moreover,

$$\begin{aligned} b(n)\lambda - a(n) &= u(n-1)\lambda - u(n) = \{\lambda(\lambda^{n-1} - \mu^{n-1}) - (\lambda^n - \mu^n)\}/2\sqrt{2} \\ &= (\lambda - \mu)\mu^{n-1}/2\sqrt{2} = \mu^{n-1} = (1 - \sqrt{2})^{n-1}. \end{aligned}$$

Thus $b(n)\lambda - a(n)$ tends to 0 and so we can apply our theorem to show that $\lambda = 1 + \sqrt{2}$ is irrational. Now if $\sqrt{2}$ were rational, we would of course have $1 + \sqrt{2}$ rational, contrary to what we have just shown. So $\sqrt{2}$ is irrational. (Fanfare.) Alternatively, a(n) = u(n) − u(n − 1) and b(n) = u(n − 1) yields a sequence of suitable fractions for $\sqrt{2}$.

There is a small but important point that we have overlooked in the previous paragraph. The theorem requires that the fractions a(n)/b(n) be in their lowest terms, that is, u(n) and u(n − 1) should have no common divisor greater than 1. We shall prove this by induction on n. Clearly u(0) = 0 and u(1) = 1 have no common divisor greater than 1. Suppose we have proved the same for u(n − 1) and u(n − 2) and let d be the greatest common divisor of u(n) and u(n − 1). Then d is also a divisor of u(n − 2) = u(n) − 2u(n − 1), from (1), so d divides both u(n − 1) and u(n − 2). By the induction assumption, we get d = 1, as required.

You might think that this was a very devious and unnecessarily complicated way to prove such a simple fact as the irrationality of $\sqrt{2}$. This may be true, but the merit of such a general "analytic" method is that it can often be applied to a much wider class of numbers than the somewhat specific (though extremely elegant) method of Pythagoras.

Quite recently the French mathematician Apéry has proved the irrationality of the number $\zeta(3)$ defined by the series

$$\zeta(3) = 1 + 1/8 + 1/27 + 1/64 + \ldots + 1/n^3 + \ldots,$$

a problem that has puzzled mathematicians since the times of Euler (the use of the letter ζ is traditional). The Swiss mathematician Leonhard Euler, one of the most famous men of science of the 18th century, proved that the number $\zeta(2)$ given by the series

$$\zeta(2) = 1 + 1/4 + 1/9 + 1/16 + \ldots + 1/n^2 + \ldots$$

is equal to $\pi^2/6$ (where π is the well known ratio of the circumference and diameter of a circle) and he found similar expressions for the sums

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