BRICKS THAT ALMOST TOPPLE OVER AGAIN

In Parabola, Volume 14, Number 3, Brendan Joyce described how to build a pile of bricks so that the top brick completely overhangs the bottom one. Now read on.

First, we shall find the formula for the maximum overhang, $H(n)$ say, which is possible with n unit bricks. Ross Baldick (Chatswood High) and Kurt Svendsen (Busby High) both sent derivations of this formula along the same lines as the one given below. Essentially, the problem reduces to a little calculation involving the principle of the lever.

As explained in our earlier article, the centre of gravity of the pile of n bricks must be directly above the edge of the table if we are to get a stable pile with the maximum overhang. Now, imagine we lift this pile of bricks and slide another brick in at the bottom. The new pile of $n+1$ bricks will be stable and have the maximum overhang if the centre of gravity of the top n bricks lies directly above the right-hand edge of the bottom brick and the centre of gravity of the whole pile lies directly above the edge of the table. The distance between the centre of gravity of the top n bricks and the centre of gravity of the whole pile is, on the one hand,

$$H(n+1) - H(n),$$

as shown in Figure 1, and on the other hand, it is

$$1/2(n+1),$$

since the bottom brick contributes $1/(n+1)$ of the total mass of the pile. So we have

$$H(n+1) = H(n) + 1/2(n+1).$$

With one brick, we can get an overhang of half a brick, so $H(1) = ½$ and, by induction, $H(n)$ is given by the sum

$$H(n) = 1/2 + 1/4 + 1/6 + \ldots + 1/2n. \qquad (1)$$

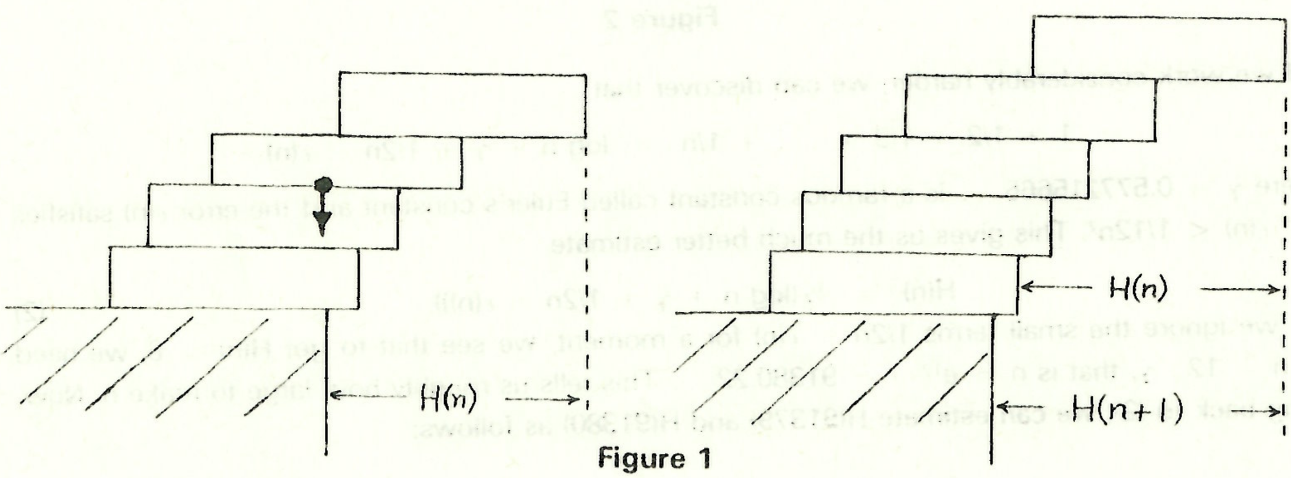

Figure 1

In particular, since the harmonic series $\Sigma 1/n$ is divergent, we can make the overhang as large as we like by using a sufficiently enormous number of bricks.

Paul Hogwood and Otis Wright (Davidson High School) wrote in response to our query about the number of bricks needed to produce an overhang of 6 bricks. After running his TI-59 calculator for a mere 20 hours, Paul found that 91380 bricks can be set up to give an overhang of 6 bricks. (He computed the exact overhang to be 6.00000142 bricks.) Otis Wright arrived at the same answer of 91380 bricks in just 26 hours and 45 minutes and reported the actual overhang to be 6.00000155 bricks. In our earlier article, we guessed that 91500 bricks would do the trick. How did we get so close?

What we have to do is to estimate the size of the sum (1) for H(n). This is good fun. By crudely comparing areas in Figure 2, we get the rough estimate

$$\log n = \int_1^n 1/x \, dx < 1 + 1/2 + 1/3 + \ldots + 1/n < 1 + \int_1^n 1/x \, dx = 1 + \log n.$$

Consequently, we have

$$H(n) = \tfrac{1}{2}(\log n + \epsilon),$$

where the error ϵ in this approximation satisfies $0 < \epsilon < 1$. If we want to make $H(n) = 6$, that is an overhang of 6 bricks, we will need to have $\log n + \epsilon = 12$. Thus, a six brick overhang will occur somewhere in the range $11 < \log n < 12$, that is $e^{11} < n < e^{12}$, which is roughly $59000 < n < 163000$. This is rather like saying that Fiji is somewhere in the Pacific Ocean.

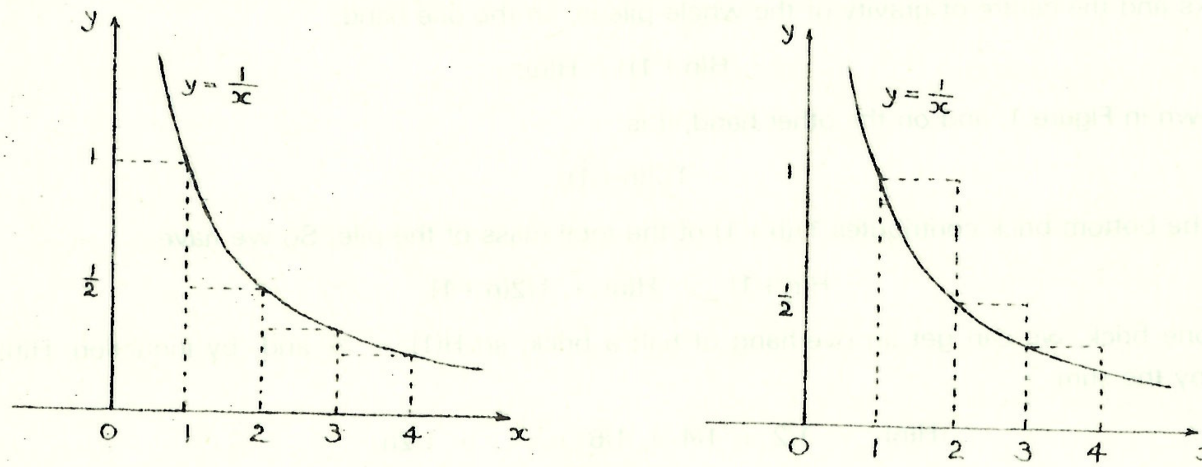

Figure 2

If we work considerably harder, we can discover that

$$1 + 1/2 + 1/3 + \ldots + 1/n = \log n + \gamma + 1/2n - \epsilon(n)$$

where $\gamma = 0.577215665\ldots$ is a famous constant called Euler's constant and the error $\epsilon(n)$ satisfies $0 < \epsilon(n) < 1/12n^2$. This gives us the much better estimate

$$H(n) = \tfrac{1}{2}(\log n + \gamma + 1/2n - \epsilon(n)). \quad (2)$$

If we ignore the small terms $1/2n - \epsilon(n)$ for a moment, we see that to get $H(n) = 6$, we need $\log n = 12 - \gamma$, that is $n = e^{12-\gamma} = 91380.22\ldots$ This tells us roughly how large to make n. Now, going back to (2), we can estimate H(91379) and H(91380) as follows:

$$H(91379) = 5.99999606, \quad H(91380) = 6.00000153.$$

(These values are actually correct to 8 decimal places, the maximum accuracy on my calculator, because the error $\epsilon(n)$ is less than 10^{-11} for these large values of n.) So we see that the number of bricks needed for a six brick overhang is 91380.

In deriving (2), we have used a sledge-hammer known as the Euler-MacLaurin summation formula. It is an extremely powerful weapon. In this case, we have used it to get the sum of the first 10^5 terms of the harmonic series correct to 8 decimal places. Suppose, instead, that we try to add these terms up on a standard calculator, working to 10 decimal places. The calculator computes each reciprocal with a possible error of, say 10^{-10} and so, after adding 10^5 terms of the series, the total error might amount to as much as 10^{-5}. This error is about as big as the last term we have tried to add on and so the sum reported by the calculator must be considered a little suspect.

If we rule out calculator fatigue and make a slightly less pessimistic error analysis, we find that the mammoth calculations reported earlier should be accurate to about 6 decimal places, which is just sufficient to determine the correct brick count.

Will anyone take on the 7 brick overhang? Otis Wright estimates that the additions would take 3 to 4 days on his calculator. But would that solve the problem?

☆ ☆ ☆

A Rational Approach to Irrational Numbers (continued from page 25)

$$\zeta(4) = \sum_{n=1}^{\infty} 1/n^4, \quad \zeta(6) = \sum_{n=1}^{\infty} 1/n^6, \text{ etc.}$$

For instance $\zeta(4) = \pi^4/96$ and, generally, $\zeta(2k)/\pi^{2k}$ is a rational number for every natural number k. Now all powers of π are known to be irrational and therefore all the numbers $\zeta(2k)$, $k = 1, 2, \ldots$ are irrational. Perhaps it is worth mentioning that the irrationality of π was first proved by Lambert, a contemporary of Euler, and the irrationality of all powers of π by Lindemann, as late as the end of the last century. Euler was unable to find a similar nice expression for $\zeta(3)$, $\zeta(5)$, etc. and it is very doubtful that such expressions exist. It caused quite a stir when Apéry succeeded in proving the irrationality of $\zeta(3)$. His method is basically the same as the one I have shown you above for $\sqrt{2}$ although details of course are far more complicated and beyond the scope of Parabola. It is nevertheless worth mentioning that the recursion used by Apéry for $\zeta(3)$ is

$$n^3u(n) = (34n^3 - 51n^2 + 27n - 5)u(n-1) + (n-1)^3u(n-2).$$

This recursion has the remarkable property that if we specify the initial values to be $u(0) = 1$ and $u(1) = 5$, then contrary to expectations, all subsequent values of u(n) are integers. No generalisation of Apéry's recursion has yet been found which would enable us for instance to prove the irrationality of $\zeta(5)$.

Trigon, where this article first appeared, is a mathematics magazine published by the Mathematical Association of South Australia. Contact: Mrs M. Wardrop, Wattle Park Teachers' Centre, 424 Kensington Road, Wattle Park, South Australia. 5066.