# PROBLEM SECTION

*You are warmly invited to submit solutions to one or more of the problems that follow. Please begin each problem on a new page and make sure that each answer bears your name, year and school. Solutions, together with the names of successful solvers, will be published in the issue after next.*

**417.** Let a and b be integers. Show that $10a+b$ is a multiple of 7 if and only if $a-2b$ is also.

**418.** Two classes organised a party. To meet the expenses, each pupil of class A paid \$5 and each pupil of class B paid \$3. If the pupils of class A had paid all the expenses, they would have paid \$k each. At a second similar event, the pupils of class A paid \$4 each and those of class B paid \$6 each, and the total sum was the same as if each pupil in class B had paid \$k. Find k. Which class had more pupils?

**419.** Write on a large blackboard the numbers 1, 2, 3, . . ., 1979. Erase any two of the numbers and replace them by their difference. Repeat this process until only a single number is left on the board. Prove that this number is even.

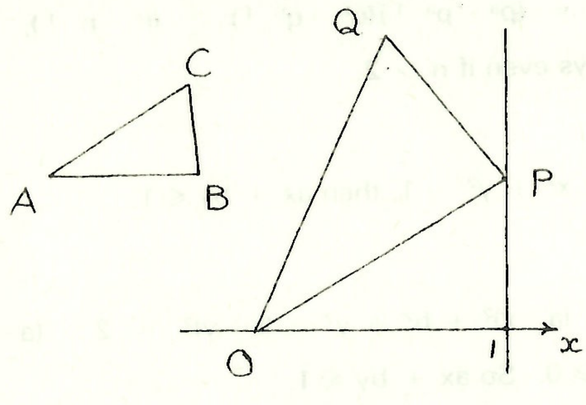
**420.** King Arthur's knights arrange a tournament. After it is all over, the king notices that to every two knights, there is a third one who has vanquished both. How many knights (at least) must have taken part in the tournament?

**421.** In the sequence 19796. . ., each digit after 6 is the last digit of the sum of the preceding four digits. (Thus, the next digit is 1 since $9+7+9+6=3\underline{1}$.) Show that . . .1979 . . . turns up again in the sequence, but that . . .1980. . . never occurs at all.

**422.** The heptagon ABCDEFG is inscribed in a circle (that is, all of its vertices A, B, . . ., G are on the circle) and three of its angles are 120°. Prove that the heptagon has two equal sides.

**423.** Given two intersecting straight lines a and b and a point P on b, show how to construct a circle whose centre is on b and which passes through P and touches a.

**424.** A triangle ABC is given in the x-y plane. Now, O is the origin, the point P moves along the line $x = 1$ and the point Q is determined so that the triangles ABC and OPQ are similar (that is, angle QOP = angle CAB and angle QPO = angle CBA). Describe the motion of Q as P moves.




**425.** Show that $2\cos x + 1 = 4\cos^2 \frac{1}{2}x - 1$. Find

$$\lim_{n \to \infty} (2\cos(x/2) - 1)(2\cos(x/2^2) - 1)\ldots(2\cos(x/2^n) - 1).$$

**426.** Find all pairs (m,n) of integers so that $x^2 + mx + n$ and $x^2 + nx + m$ both have integer roots. (For example $x^2 + 5x + 6 = (x+2)(x+3)$ and $x^2 + 6x + 5 = (x+1)(x+5)$.)

**427.** The four aces, kings, queens and jacks are taken from a pack of cards and dealt to four players. Thereupon, the bank pays $1 for every jack held, $3 for every queen, $5 for every king and $7 for every ace. In how many ways can it happen that all four players receive equal payments (namely $16)?

**428.** Let n be an integer whose last digit is 7. Show that some multiple of n has no digit equal to zero.

## SOLUTIONS TO PROBLEMS FROM VOLUME 14, NUMBER 3

**393.** Show that if n is any integer greater than 2, of the fractions $1/n, 2/n, 3/n, \ldots, (n-1)/n$ an even number are in lowest terms.

**Solution I.**

Suppose $0 < h < \frac{1}{2}n$. The fraction $h/n$ is in lowest terms if and only if $(n-h)/n$ is in lowest terms. (Why?) Thus the fractions in lowest terms can be "paired off", each one less than ½ being paired with one greater than ½. (For example, if $n = 8$, then 1/8 is paired with 7/8 and 3/8 with 5/8.) Consequently, the number of the fractions $1/n, 2/n, \ldots, (n-1)/n$ in lowest terms is even. The argument breaks down if $n = 2$, since then the fraction 1/2 is in lowest terms and is left "unpaired".