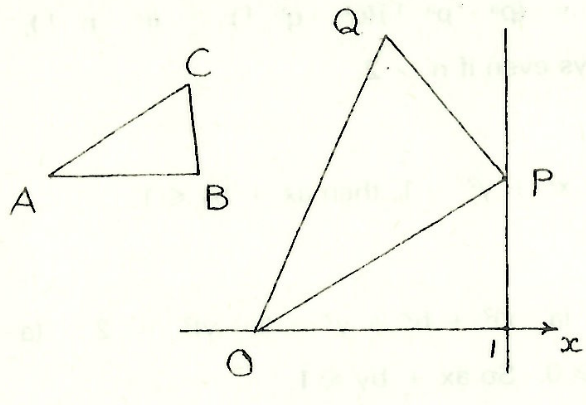
**424.** A triangle ABC is given in the x-y plane. Now, O is the origin, the point P moves along the line $x = 1$ and the point Q is determined so that the triangles ABC and OPQ are similar (that is, angle QOP = angle CAB and angle QPO = angle CBA). Describe the motion of Q as P moves.




**425.** Show that $2\cos x + 1 = 4\cos^2 \frac{1}{2}x - 1$. Find

$$\lim_{n \to \infty} (2\cos(x/2) - 1)(2\cos(x/2^2) - 1) \ldots (2\cos(x/2^n) - 1).$$

**426.** Find all pairs (m,n) of integers so that $x^2 + mx + n$ and $x^2 + nx + m$ both have integer roots. (For example $x^2 + 5x + 6 = (x+2)(x+3)$ and $x^2 + 6x + 5 = (x+1)(x+5)$.)

**427.** The four aces, kings, queens and jacks are taken from a pack of cards and dealt to four players. Thereupon, the bank pays \$1 for every jack held, \$3 for every queen, \$5 for every king and \$7 for every ace. In how many ways can it happen that all four players receive equal payments (namely \$16)?

**428.** Let n be an integer whose last digit is 7. Show that some multiple of n has no digit equal to zero.

## SOLUTIONS TO PROBLEMS FROM VOLUME 14, NUMBER 3

**393.** Show that if n is any integer greater than 2, of the fractions 1/n, 2/n, 3/n, . . ., (n − 1)/n an even number are in lowest terms.

**Solution I.**

Suppose $0 < h < \frac{1}{2}n$. The fraction h/n is in lowest terms if and only if (n − h)/n is in lowest terms. (Why?) Thus the fractions in lowest terms can be "paired off", each one less than ½ being paired with one greater than ½. (For example, if n = 8, then 1/8 is paired with 7/8 and 3/8 with 5/8.) Consequently, the number of the fractions 1/n, 2/n, . . ., (n − 1)/n in lowest terms is even. The argument breaks down if n = 2, since then the fraction 1/2 is in lowest terms and is left "unpaired".

**Solution II from Ross Baldick (Chatswood High School).**

The problem is the same as finding the number of integers less than or equal to n which are relatively prime to n; this number is denoted by $\phi(n)$ and called Euler's function. If n has the prime factorisation $n = p^a q^b \ldots r^c$, then we have the formula

$$\phi(n) = (p^a - p^{a-1})(q^b - q^{b-1}) \ldots (r^c - r^{c-1}),$$

and it follows that $\phi(n)$ is always even if $n > 2$.

**394.** Prove that, if $a^2 + b^2 = x^2 + y^2 = 1$, then $ax + by \leqslant 1$.

**Solution I.**

$$2ax + 2by = a^2 + x^2 - (a-x)^2 + b^2 + y^2 - (b-y)^2 = 2 - (a-x)^2 - (b-y)^2 \leqslant 2,$$

since $(a-x)^2 \geqslant 0$ and $(b-y)^2 \geqslant 0$. So $ax + by \leqslant 1$.

**Solution II from Ross Baldick (Chatswood High School) and K. Svendsen (Busby High School).**

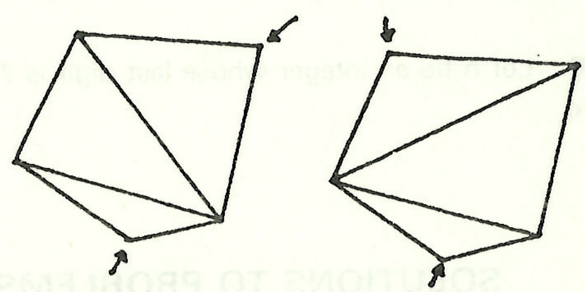
In view of the identity, $\sin^2\theta + \cos^2\theta = 1$, we may assume $a = \sin\theta$, $b = \cos\theta$, $x = \sin\phi$, $y = \cos\phi$. Then

$$ax + by = \sin\theta\sin\phi + \cos\theta\cos\phi = \cos(\theta - \phi) \leqslant 1.$$

**395.** A polygon is said to be triangulated when diagonals, no two of which cross, are drawn cutting the polygon into triangles. A polygon other than a triangle can be triangulated in more than one way.

(a) Show that a triangulated n-gon is always cut into $n-2$ triangles by $n-3$ diagonals.

(b) Show that there are at least two vertices of a triangulated polygon each of which lies in a single triangle.



**Solution I.**

A proof by mathematical induction can be given for both parts.

(a) The statement is trivial when $n = 3$. Assume it is true when $n \leqslant k-1$, that is every triangulation of such an n-gon has $n-3$ diagonals and $n-2$ triangles. Now consider one diagonal in any triangulation of a k-gon. This divides the figure into an r-gon (with $r \geqslant 3$) and a $(k+2-r)$-gon. The remaining diagonals triangulate both of these. By the induction hypothesis, the total number of triangles is $(r-2) + (k+2-r-2) = k-2$ and the total number of diagonals is $1 + (r-3) + (k+2-r-3) = k-3$. This completes the proof.

(b) Again the statement is trivial when $n = 3$, but it is convenient to prove the slightly stronger statement that, when $n \geqslant 4$, there are at least two *non-adjacent* vertices of the triangulated polygon each of which lies in a single triangle.

This assertion is readily checked when $n = 4$, for each of the two possible triangulations. Assume it is true for all $n \leqslant k-1$ and consider a triangulated k-gon. As in (a), one diagonal, AB say, of the triangulation divides the figure into two poygons, each of which is triangulated by the remaining diagonals. By the induction hypothesis, in each of these two polygons there are two vertices each lying in a single triangle, and since they are non-adjacent, they are not at both ends of the edge AB, that is at least one of them occurs at a vertex other than A or B. Thus there are at least two non-adjacent vertices of the triangulated k-gon each in a single triangle. This completes the proof.

**Solution II from Ross Baldick (Chatswood High School).**

Here is a nicer proof of (b). By (a), the triangulated n-gon has n sides and $n-2$ triangles. So there are at least two triangles in the triangulation, each of which contains two sides of the n-gon. In each case, the two sides in question are necessarily adjacent and no diagonal of the triangulation can end at their common vertex, so this vertex lies in a single triangle.

**396.** Find a 10 digit number whose first digit tells the number of zeros which appear in it, whose second digit tells the number of ones, and so on; (thus the tenth digit tells the number of nines in the number). Is there another such number?

**Solution.**

The only such number is 6210001000.

Let a(0) a(1) a(2) . . . a(9) be a number with the given properties. Then

$$a(0) + a(1) + a(2) + \ldots + a(9) = 10 \quad (1)$$

and

$$0.a(0) + 1.a(1) + 2a(2) + \ldots + 9a(9) = 10. \quad (2)$$

The first equation counts the number of digits in the number by adding the number of zeros, of ones, and so on, and says that the sum of the digits is 10. The second equation adds the digits in a different way by summing all the zeros, all the ones, and so on, and adding the totals. Subtracting (2) from (1) gives

$$a(0) = a(2) + 2a(3) + \ldots + 8a(9). \quad (3)$$

If $a(0) = k$, say, then $a(k) \geqslant 1$ and one term on the right side of (3) is $(k-1)a(k)$. Hence we must have $a(k) = a(2) = 1$ and $a(j) = 0$ for $j > 2$ and $j \neq k$. Now (3) reads

$$k = 1 + 0 + 0 + \ldots + (k-1).1 + \ldots + 0.$$

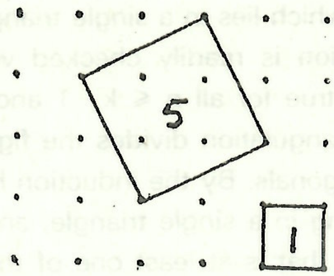
Consequently, $a(1) = 2$, $a(2) = a(k) = 1$ and, from (1), $a(0) = 6$.

Correct solutions were received from T. Abberton (St. Paul's College, Bellambi), K. Svendsen (Busby High School), Surinder Wadhwa (Ashfield Boys' High School) and Richard Wilson (The King's School, Parramatta).

**397.** The smallest square on a peg-board has unit area. The figure shows how to construct squares of area 1 and 5 using pegs and rubber bands.

(a) Show how to construct squares of areas 8 and 10.

(b) Prove that it is not possible to construct a square of area $4n + 3$ where n is an integer.

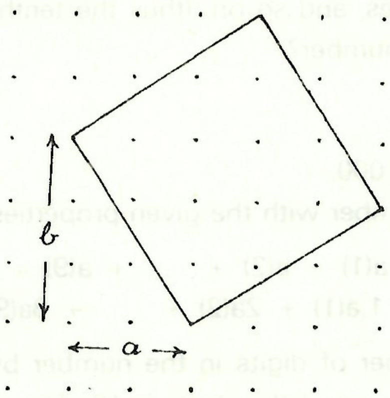



**Solution from Ross Baldick (Chatswood High School) and Richard Wilson (The King's School, Parramatta).**

(a) By Pythagoras' theorem, the area A of the square in the figure is $a^2 + b^2$. We have $A = 8$ if $a = b = 2$, and $A = 10$ if $a = 3$, $b = 1$.

(b) The square of an even number is a multiple of 4 and the square of an odd number is one more than a multiple of 4, so the remainder when $a^2 + b^2$ is divided by 4 is 0 (when a and b are both even), or 1 (when a is even and b is odd, or vice versa), or 2 (when a and b are both odd), but is never 3.

A correct solution was also received from K. Svendsen (Busby High School) and partial solutions were received from Jennifer Taylor (Woy Woy High School) and Surinder Wadhwa (Ashfield Boys' High School).




398. Show that it is impossible to construct an equilateral triangle on the pegboard in question 397 using 3 pegs and a rubber band.

**Solution I.**

Suppose an equilateral triangle could be constructed; let $A = (0,0)$, $B = (a,b)$ and $C = (h,k)$ be the vertices of such a triangle of minimum size. Now a, b, h and k cannot all be even, since then $(0,0)$, $(\frac{1}{2}a, \frac{1}{2}b)$, $(\frac{1}{2}h, \frac{1}{2}k)$ would be a smaller equilateral triangle. Equating the lengths of the sides of triangle ABC gives

$$a^2 + b^2 = h^2 + k^2 = (a-h)^2 + (b-k)^2. \quad (1)$$

We consider the remainders of these quantities on division by 4, as in the solution to problem 397. If a and b are both odd, then h and k must both be odd, but now $a - h$ and $b - k$ are both even, so that (1) cannot hold. If just one of a and b is odd, then also just one of h and k is odd, so $a - h$

and b − k are both odd or both even and again (1) is impossible. Finally if a and b are both even, then so are h and k and this has already been ruled out by our choice of the triangle ABC.

**Solution II from Ross Baldick (Chatswood High School).**

Again suppose we have an equilateral triangle with vertices $A = (0,0)$, $B = (a,b)$ and $C = (h,k)$, where a, b, h and k are integers. We may suppose the order of the points A, B, C round the triangle is anticlockwise. Now, by a little bit of trigonometry, we find $h = -\frac{1}{2}a - \frac{1}{2}b\sqrt{3}$ and $k = -\frac{1}{2}b + \frac{1}{2}a\sqrt{3}$. Since at least one of a and b is non-zero, these equations imply that $\sqrt{3}$ is rational, a well-known contradiction.

Partial solutions were received from K. Svendsen (Busby High School) and Richard Wilson (The King's School, Parramatta).

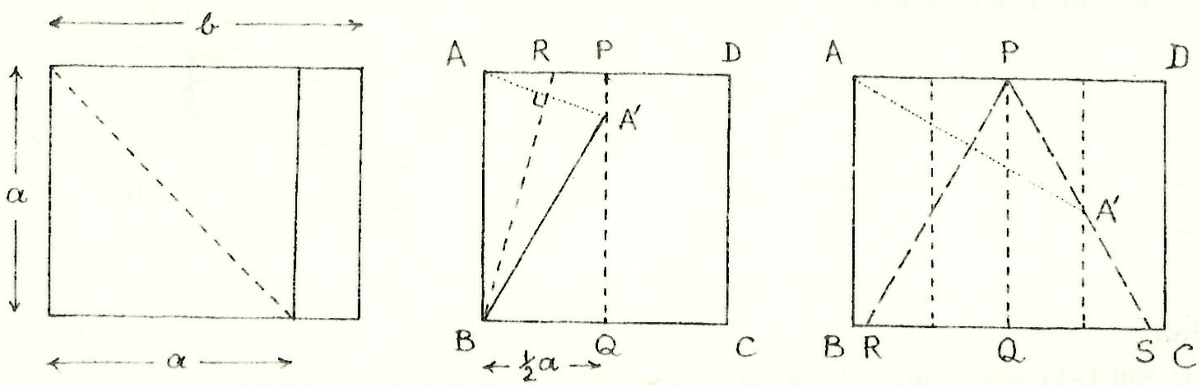
**399.** Show how to construct an equilateral triangle by folding a single (rectangular) sheet of paper. No rulers, compasses or separate sheets for measuring are to be used.

**Solution I from K. Svendsen (Busby High School).**

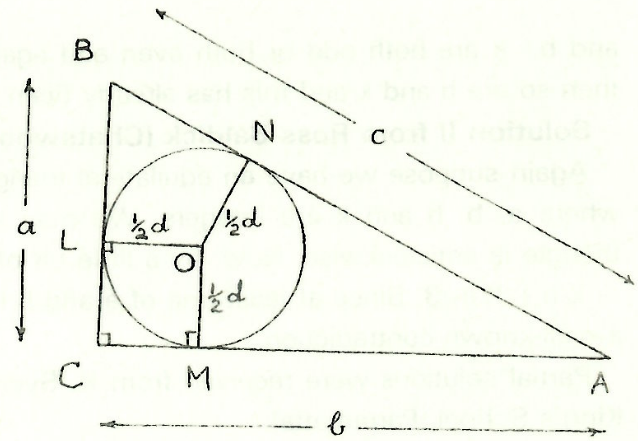
Suppose we start with an a by b rectangular sheet of paper with $a \leqslant b$. Fold the paper as indicated in the first figure to obtain an a by a square. Now, fold this square as shown in the second figure. We get the line PQ by folding the square in half, then we get the line BR by bringing the corner A to a point A′ on PQ. This makes $BA' = BA = a$ and $BQ = \frac{1}{2}a$, so A′BC is an equilateral triangle.

**Solution II from Ross Baldick (Chatswood High School).**

First fold the paper into quarters as shown in the third figure. Then make a fold PR by bringing the corner A to a point A′ on the ¾-fold at the other end. Do the same with D. The two folds PR and PS are at 60° to each other, and PRS is an equilateral triangle. (Why?) This construction works with normal sheets of paper, provided they are not too long and thin.



**400.** Show that the diameter d of the inscribed circle of a right triangle of legs a, b and hypotenuse c satisfies d = a + b − c.




**Solution from K. Svendsen (Busby High School) and Richard Wilson (The King's School, Parramatta).**

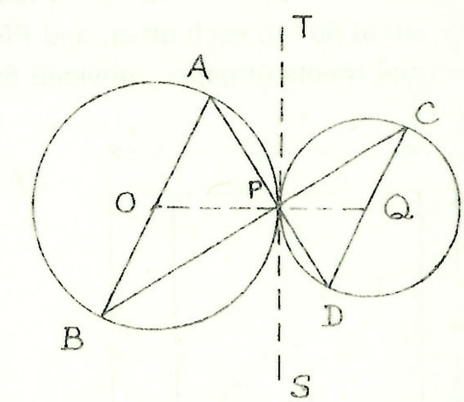
Let OL, OM and ON be the radii of the inscribed circle to the points of contact with the sides, as shown in the figure. Since the angles at L, M and C are right angles and OL = OM, we see that CMOL is a square and CL = CM = ½d. Hence BL = a − ½d and AM = b − ½d. Now the right-angled triangles OAM and OAN have a common hypotenuse OA and corresponding equal sides ON = OM, so they are congruent and AN = AM = b − ½d. Similarly BN = BL = a − ½d. So

$$c = AB = AN + BN = (a - \tfrac{1}{2}d) + (b - \tfrac{1}{2}d) = a + b - d,$$

as required.

Correct solutions were also received from Ross Baldick (Chatswood High School) and Surinder Wadhwa (Ashfield Boys' High School).

**401.** Let AB and CD be parallel diameters of two circles which touch at P. Show that the lines BC and AD intersect at P.




**Solution.**

Let O and Q be the centres of the circles and SPT be the common tangent at P. Since OPT and QPT are both right angles, OPQ is a straight line. Draw AP and DP. Now

angle APT = ½ angle AOP = ½ angle DQP = angle DPS,

so APD is a straight line. Similarly, BPC is a straight line and the result follows.

Correct solutions were received from Ross Baldick (Chatswood High School) and K. Svendsen (Busby High School).

**402.** Consider 5 points in space such that each pair is not more than 1 cm apart. What is the greatest number of pairs which can be exactly 1 cm apart? Prove your answer. (If there are 4 points there can be as many as six pairs exactly 1 cm apart — take the four points at the vertices of a regular tetrahedron).

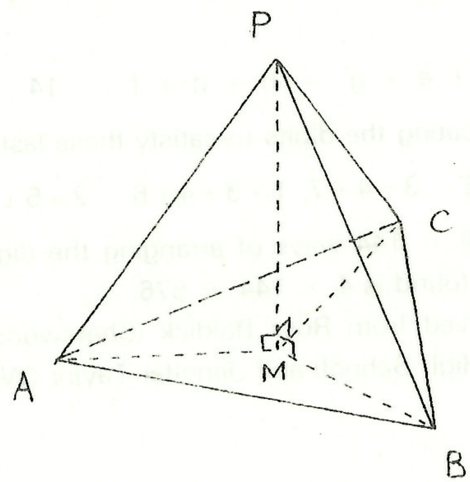
**Solution.**

Of the 10 possible pairs of points, no more than 8 can be exactly 1 cm apart. To see this, start by taking hree of the points A, B and C at the vertices of an equilateral triangle of side 1 cm. Suppose P is equidistant from A, B and C and let M be the foot of the perpendicular from P to the plane of A, B and C. By Pythagoras' theorem,

$$PM^2 + AM^2 = PM^2 + BM^2 = PM^2 + CM^2,$$

so M is the circumcentre of the triangle ABC and, in fact, $AM = \sqrt{3}/3$ cm. Consequently, we get $PA = PB = PC = 1$ cm provided $PM = \sqrt{6}/3$ cm. Thus to achieve 9 distances equal to 1 cm, we must place D at a point $\sqrt{6}/3$ cm perpendicularly above the circumcentre of triangle ABC and E at the same distance perpendicularly below the circumcentre. But the tenth distance DE is then $2\sqrt{6}/3 > 1$ cm, contrary to the conditions of the problem. So we can only make the fifth point E have distance 1 cm away from 2 of the points A, B, C and D and this gives 8 distances equal to 1 cm.

A correct solution was received from Ross Baldick (Chatswood High School) and an almost correct one from K. Svendsen (Busby High School).




**403.** Given any set of ten distinct positive integers each less than 100 show that there are two subsets of this set having no elements in common such that the sums of the numbers in the subsets are equal.

**Solution from Ross Baldick (Chatswood High School).**

Let S be a set of 10 distinct positive integers less than 100. There are $2^{10} = 1024$ ways of choosing a subset T of S (or 1023 if we discount the empty subset, having no elements). To see this, note that to choose T, we can take each of the 10 elements of S in turn and do one of 2 things with it, namely decide to put it in T, or to leave it out of T. Now, each such subset T contains at most 10 integers all less than 100, so the sum of its elements, sum (T) say, is always an integer less than 1000. Since there are 1023 > 1000 possible subsets of S, we can find two subsets, A and B say, with sum (A) = sum (B). Let A' and B' be obtained from A and B by omitting any common elements. Then both sums are decreased by the same amount, namely by sum $(A \cap B)$, so we have sum (A') = sum (B'), as required.

This is a very pretty argument. It illustrates the use of the pigeon-hole principle, which may be expressed as follows: If you try to put a flock of pigeons into a number of boxes and there are more pigeons than boxes, then there will be at least one box containing more than one pigeon. This simple idea has some far-reaching applications.

404. The number 1234567 is not divisible by 11, but 3746512 is. How many different multiples of 11 can be obtained by appropriately ordering these digits?

**Solution from Richard Wilson (The King's School, Parramatta).**

If abcdefg is divisible by 11, then

$$a - b + c - d + e - f + g = 11n, \qquad (1)$$

say, using the well-known test for divisibility by 11. For this problem,

$$a + b + c + d + e + f + g = 28 \qquad (2)$$

Adding (1) and (2) gives $2(a + c + e + g) = 28 + 11n$. But $10 \leqslant a + c + e + g \leqslant 22$, so we must have $n = 0$ and

$$a + c + e + g = b + d + f = 14.$$

There are four different ways of allocating the digits to satisfy these last two conditions, namely

$1 + 2 + 4 + 7 = 3 + 5 + 6$, $1 + 2 + 5 + 6 = 3 + 4 + 7$, $1 + 3 + 4 + 6 = 2 + 5 + 7$, $2 + 3 + 4 + 5 = 1 + 6 + 7$.

For each of these, there are $4! \times 3! = 144$ ways of arranging the digits, so the total number of different multiples of 11 that can be found is $4 \times 144 = 576$.

A correct solution was also received from Ross Baldick (Chatswood High School) and partial solutions from K. Svendsen (Busby High School) and Jennifer Taylor (Woy Woy High School).

**Solvers of earlier problems.**

The following contributions were received too late for acknowledgement in the last issue:

Ross Baldick (Chatswood High School): solutions to problems 382 to 386 and 388 to 390, all excellent;

Hee Chan (Crows Nest Boys' High School): a solution to problem 382 and a partial solution to problem 386;

Paul Rider (St. Leo's College): a solution to problem 382.

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