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MATRICES CAN BE USEFUL – AND EVEN FUN!

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Introduction

Let me begin by describing some of the basic properties of matrices. A *matrix* is a rectangular array of elements, usually numbers, for example,

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1.3 & 1.2 \\ 1.7 & 0.6 \\ 0.2 & 1.3 \end{bmatrix}, \quad \begin{bmatrix} 4.3 & 0.0 \\ 1.9 & 0.3 \end{bmatrix}.$$

A matrix with m rows and n columns is called an $m \times n$ matrix. The sizes of the above matrices are 2×1, 3×2, and 2×2 respectively. A matrix of size $n \times 1$ is called a (column) *vector.*

For two matrices to be *equal,* they must have the same number of elements arranged in exactly the same pattern, and have the same elements in the same places. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}.$$

If two matrices A and B have the same size, the *sum* $A+B$ of the two matrices is the matrix obtained by adding the corresponding elements of A and B. Thus, if

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} -3 & 2 \\ 4 & 1 \end{bmatrix}, \text{ then } A+B = \begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix}.$$

In a similar manner,

$$A-B = \begin{bmatrix} 4 & 0 \\ -7 & 3 \end{bmatrix}.$$

The sum and difference are not defined when A and B are not of the same size. A matrix with all its elements zero is called a *zero matrix* and is denoted by O. A matrix can be multiplied by a constant by multiplying each of its elements by the constant. Thus

$$-3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ -9 & -12 \end{bmatrix} = 3\begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}.$$

The product of two matrices is more difficult. The *product* AB of the two matrices A and B exists if the number of columns of A is equal to the number of rows of B. The resulting product matrix

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AB has the same number of rows as A and the same number of columns as B. The element in the r-th row and the s-th column of AB is obtained by multiplying each element in the r-th row of A by the corresponding element in the s-th column of B and then adding these products. For example, the element in the second row and first column of the product

$$\text{second row} \rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 1 & 9 & 2 \\ 4 & 7 & 6 \\ 14 & 12 & 13 \end{bmatrix} \begin{bmatrix} -1 & -5 \\ 12 & 11 \\ 8 & 0 \end{bmatrix}$$

$$\uparrow \text{ first column}$$

is $[1 \times (-1)] + [9 \times 12] + [2 \times 8] = 123$. In this case, the first matrix is 4×3, the second is 3×2, and the product matrix is 4×2. It takes the form

$$\begin{bmatrix} 2\times(-1) + 3\times 12 + 5\times 8 & 2\times(-5) + 3\times 11 + 5\times 0 \\ 1\times(-1) + 9\times 12 + 2\times 8 & 1\times(-5) + 9\times 11 + 2\times 0 \\ 4\times(-1) + 7\times 12 \times 6\times 8 & 4\times(-5) + 7\times 11 + 6\times 0 \\ 14\times(-1) + 12\times 12 + 13\times 8 & 14\times(-5) + 12\times 11 + 13\times 0 \end{bmatrix} = \begin{bmatrix} 74 & 23 \\ 123 & 94 \\ 128 & 57 \\ 234 & 62 \end{bmatrix}$$

This method of defining a matrix product may seem complicated, but it is extremely useful, because we often meet sums of products of pairs of numbers. We shall give some examples later. Note that in the above example, the product BA is not defined. Even when AB and BA are both defined, the two products are not, in general, equal. A square matrix with ones down the main diagonal and zeros elsewhere is called a *unit matrix* and is denoted by I. For example, the 3×3 unit matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is easy to verify that whenever the products are defined $AI = A = IA$. A square matrix has an inverse if there is a matrix A^{-1} such that $AA^{-1} = I = A^{-1}A$, and then A^{-1} is called the *inverse* of A.

An interesting trick

The matrix $\begin{bmatrix} 1 & ½ \\ ½ & 1 \end{bmatrix}$ can be written as $I + A$ with $A = \begin{bmatrix} 0 & ½ \\ ½ & 0 \end{bmatrix}$. We note that

$$A^2 = \begin{bmatrix} ¼ & 0 \\ 0 & ¼ \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 1/8 \\ 1/8 & 0 \end{bmatrix}, \quad A^4 = \begin{bmatrix} 1/16 & 0 \\ 0 & 1/16 \end{bmatrix}, \quad A^5 = \begin{bmatrix} 0 & 1/32 \\ 1/32 & 0 \end{bmatrix}, \text{ etc.}$$

If we pretend that A is an ordinary number instead of a matrix, we might be tempted to write

$$(I + A)^{-1} = I - A + A^2 - A^3 + A^4 - \ldots .$$

Let us see whether this is true for a matrix. The right-hand side is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & ½ \\ ½ & 0 \end{bmatrix} + \begin{bmatrix} ¼ & 0 \\ 0 & ¼ \end{bmatrix} - \begin{bmatrix} 0 & 1/8 \\ 1/8 & 0 \end{bmatrix} + \begin{bmatrix} 1/16 & 0 \\ 0 & 1/16 \end{bmatrix} - \begin{bmatrix} 0 & 1/32 \\ 1/32 & 0 \end{bmatrix} + \ldots$$

We note that $1 + 1/4 + 1/16 + \ldots = 1/(1 - ¼) = 4/3$ and $1/2 + 1/8 + 1/32 + \ldots = ½/(1 - ¼) = 2/3$, and so the matrices written out above sum to

$$\begin{bmatrix} 1 + 1/4 + 1/16 + \ldots & -1/2 - 1/8 - 1/32 - \ldots \\ -1/2 - 1/8 - 1/32 - \ldots & 1 + 1/4 + 1/16 + \ldots \end{bmatrix} = \begin{bmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{bmatrix}.$$

Is this matrix the inverse of $(I + A)$? We can check as follows:

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4/3 & -2/3 \\ -2/3 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}.$$

So the trick of summing the matrix geometric progression works (at least in this particular case).

Eigenvalues and eigenvectors

A square matrix A has an *eigenvalue* λ with corresponding *eigenvector* **x** if $A\mathbf{x} = \lambda\mathbf{x}$. Take, for example,

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so 5 and -1 are eigenvalues of A and the corresponding eigenvectors are $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively. Note that

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}^3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5^2 \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5^3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

In general,

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 5^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and similarly} \quad \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Now consider an arbitrary column vector, say $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and let us calculate $\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}^4 \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. We write the given vector as a linear combination of the eigenvectors as follows

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

By our earlier observations, we get

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}^4 \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}^4 \left\{ 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} = 2 \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}^4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}^4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \times 5^4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-1)^4 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The second term is relatively small compared to the first and, omitting it, we obtain

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}^4 \begin{bmatrix} 5 \\ 1 \end{bmatrix} \approx 2 \times 5^4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2500 \\ 1250 \end{bmatrix}.$$

The exact answer can of course be obtained by direct multiplication and it is $\begin{bmatrix} 2501 \\ 1249 \end{bmatrix}$.

The important thing to notice is that when an arbitrary vector is pre-multiplied by a high power of a matrix, the resultant vector is approximately proportional to the dominant eigenvector, that is the eigenvector belonging to the largest eigenvalue.

The matrix approach to population analysis

A species of animal lives for three years only. Three-quarters of the females born survive to one year of age and two-thirds of those reaching age one survive to age two. On average, each female reaching age one has one daughter before attaining age two and each female reaching age two has 65/32 daughters before age three.

Let us denote the number of females in year t whose age last birthday was x by n(x,t). In terms of the n(x,t), the problem becomes

$$n(0,t+1) = n(1,t) + (65/32)\,n(2,t),\quad n(1,t+1) = \tfrac{3}{4}\,n(0,t),\quad n(2,t+1) = \tfrac{2}{3}\,n(1,t).$$

These linear recurrence equations between the n(x,t) and n(x,t + 1) can be written in matrix notation as follows:

$$\begin{bmatrix} n(0,t+1) \\ n(1,t+1) \\ n(2,t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 65/32 \\ 3/4 & 0 & 0 \\ 0 & 2/3 & 0 \end{bmatrix} \begin{bmatrix} n(0,t) \\ n(1,t) \\ n(2,t) \end{bmatrix}$$

or, more concisely, as $\mathbf{n}(t+1) = A\mathbf{n}(t)$. From this, we see that the solution of the recurrence is

$$\mathbf{n}(t) = A^t\,\mathbf{n}(0).$$

The matrix A just introduced has the dominant eigenvalue 5/4 and the corresponding eigenvector is

$$\mathbf{x} = \begin{bmatrix} 25 \\ 15 \\ 8 \end{bmatrix}$$

(that is $A\mathbf{x} = (5/4)\mathbf{x}$). The initial age vector $\mathbf{n}(0)$ will have a component $k\mathbf{x}$, say, in the $\mathbf{x}$-direction. We can invoke the argument of the previous section to show that, for large t,

$$\mathbf{n}(t) = A^t\,\mathbf{n}(0) \approx (5/4)^t\,k\mathbf{x}, \qquad \text{that is} \qquad \begin{bmatrix} n(0,t) \\ n(1,t) \\ n(2,t) \end{bmatrix} \approx (5/4)^t\,k \begin{bmatrix} 25 \\ 15 \\ 8 \end{bmatrix}$$

Thus, irrespective of its initial age structure, the population grows asymptotically at 25% per annum and it adopts a stable age structure with 25/48 of the females aged zero, 15/48 aged one and 8/48 aged two.

No-claim discounts in motor-vehicle insurance

Consider a motor-vehicle insurance company starting business and writing a thousand new policies each year. All new policy-holders pay the full premium. Motorists with the company having only the previous year free of claim receive a discount of 30%, while those with the two preceding years free of claim receive a discount of 50%. A motorist making a claim in a particular year pays the full premium the following year. Let us denote the number of policy-holders with no discount, 30% discount and 50% discount in year t of the company's operations by n(1,t), n(2,t) and n(3,t) respectively. We further assume that 10% of the motorists insuring with the company (irrespective of their discount position) claim in a year and a further 10% fail to renew their policies. All this information gives us the equations

$$n(1,t+1) = 0.1\ n(1,t) + 0.1\ n(2,t) + 0.1\ n(3,t) + 1000$$
$$n(2,t+1) = 0.8\ n(1,t)$$
$$n(3,t+1) = 0.8\ n(2,t) + 0.8\ n(3,t).$$

In matrix notation, this becomes

$$\begin{bmatrix} n(1,t+1) \\ n(2,t+1) \\ n(3,t+1) \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.8 & 0 & 0 \\ 0 & 0.8 & 0.8 \end{bmatrix} \begin{bmatrix} n(1,t) \\ n(2,t) \\ n(3,t) \end{bmatrix} + \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

or, more concisely, $\mathbf{n}(t+1) = A\ \mathbf{n}(t) + \mathbf{b}$. Applying this equation recursively, we have

$$\mathbf{n}(1) = A\ \mathbf{n}(0) + \mathbf{b} = \mathbf{b} \text{ (since } \mathbf{n}(0) = \mathbf{0}),$$
$$\mathbf{n}(2) = A\ \mathbf{n}(1) + \mathbf{b} = (A+I)\ \mathbf{b},$$
$$\mathbf{n}(3) = A\ \mathbf{n}(2) + \mathbf{b} = (A^2+A+I)\ \mathbf{b},$$

and in general

$$\mathbf{n}(t) = (I + A + A^2 + \ldots + A^{t-1})\ \mathbf{b} = (I - A^t)(I - A)^{-1}\ \mathbf{b}.$$

(Recall the identity $1 + x + x^2 + \ldots + x^{t-1} = (1 - x^t)/(1 - x)$ and our earlier observations on summing matrix geometric progressions.)

In this case, the dominant eigenvalue of A is less than one, and this means that A^t tends to zero as t tends to infinity. It follows that, for large t,

$$\mathbf{n}(t) \approx (I - A)^{-1}\ \mathbf{b} = \begin{bmatrix} 0.9 & -0.1 & -0.1 \\ -0.8 & 1.0 & 0 \\ 0 & -0.8 & 0.2 \end{bmatrix}^{-1} \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1.6 & 1.8 & 0.8 \\ 6.4 & 7.2 & 8.2 \end{bmatrix} \begin{bmatrix} 1000 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2000 \\ 1600 \\ 6400 \end{bmatrix}.$$

In other words, the insurance company's portfolio approaches a stationary position with 2000 policy-holders paying the full premium, 1600 on 30% discount and 6400 on 50% discount. The total number of policy-holders approaches 10000 with an average discount of 36.8%.

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