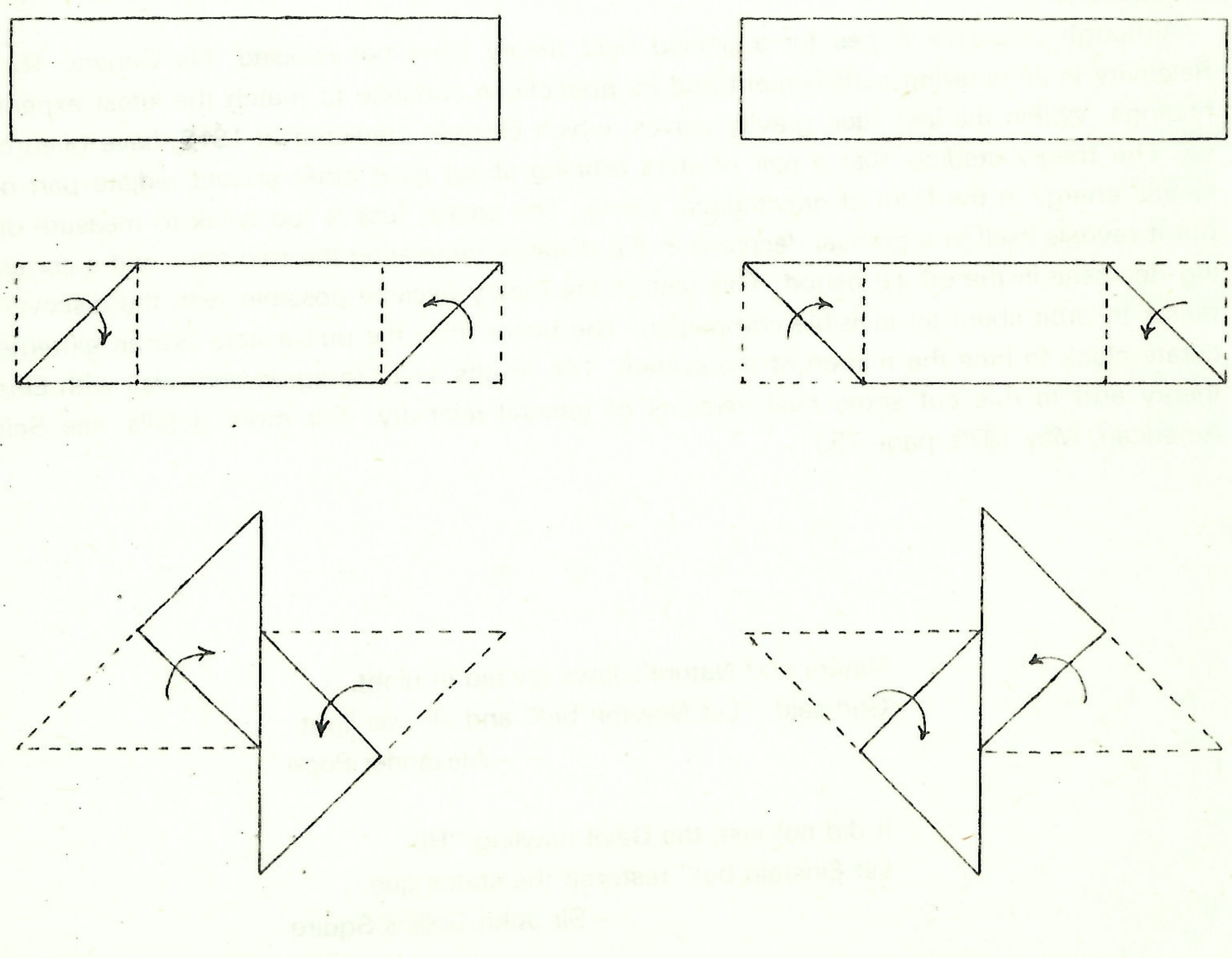
LETTERS TO THE EDITOR

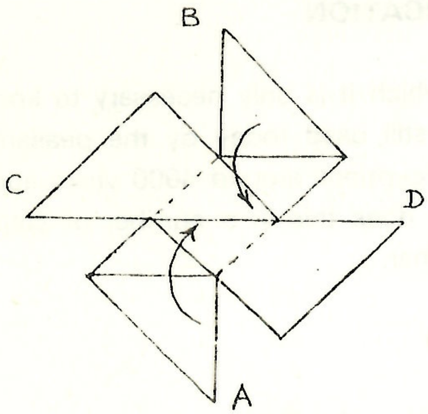
MAGIC STARS

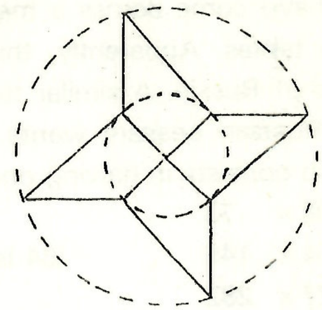
Dear Sir,

In answer to a question asked by Julian Abel in Parabola, Volume 15, Number 1, I am sending you a four-pointed star. The construction uses two circles, as shown in the last figure opposite. (Actually, it is not possible to draw a four-pointed star by the construction given in Julian Abel's article.)

Here is a method for making a four-pointed star by Origami. Start with two rectangular pieces of paper each measuring 4 units by 1 unit and fold them as shown in the first four figures. Turn the first one over and place the second on top of it as in the fifth figure. Fold and tuck in the corners A and B. Then turn the whole thing over and fold and tuck in the corners C and D. And there you are! Did it work?







I have found by experiment that the sum of the first n odd numbers is equal to n^2. For example,

$$1 + 3 = 2^2, \quad 1 + 3 + 5 = 3^2, \quad 1 + 3 + 5 + 7 = 4^2.$$

Would you explain why this works.

Jennifer Taylor,
Year 10,
Woy Woy High School.

Editor's comments.

Here are two ways to find the sum of the first n odd numbers:

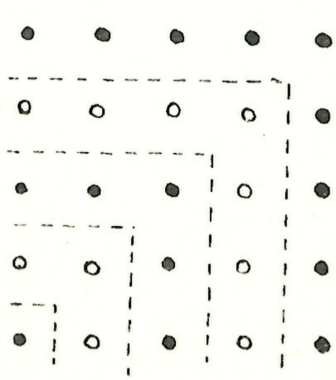
$$s(n) = 1 + 3 + 5 + \ldots + (2n-5) + (2n-3) + (2n-1).$$

For the first method, suppose we pair the numbers in the sum by adding the first and last terms, the second and second last, and so on:

$$s(n) = [1 + (2n-1)] + [3 + (2n-3)] + [5 + (2n-5)] + \ldots$$

If n is even, we get $\frac{1}{2}n$ pairs and the sum of each pair is $2n$, so we have $s(n) = \frac{1}{2}n \times 2n = n^2$. If n is odd, we get $\frac{1}{2}(n-1)$ pairs each with sum $2n$ and one number, namely n, left over in the middle. So, again, the sum is $s(n) = \frac{1}{2}(n-1) \times 2n + n = n^2$. Try this out with $n = 4$ or 5 to see why it works. (This method can be used to sum any arithmetic progression. Can you see how?)

The second method is geometrical. We represent the numbers 1, 3, 5, ... by dots and arrange them to form an $n \times n$ square. For example, the figure shows at a glance that $1+3+5+7+9 = 5^2$. You will find more of these geometrical ideas for summing series in Martin Gardner's column in Scientific American, October 1973.



$1 + 3 + 5 + 7 + 9 = 5^2$

RUSSIAN MULTIPLICATION

Dear Sir,

I have come across a method of multiplication for which it is only necessary to know the two time tables. Apparently, this very economical idea is still used today by the peasants in some parts of Russia. A similar technique was used by the Egyptians around 4000 years ago. Suppose the Russian peasant wants to multiply 108 by 73. He does this in a number of steps, each of which consists in halving one factor and doubling the other.

	108×73	
	54×146	(54 is half of 108; $146 = 2 \times 73$)
*	27×292	
*	13×584	(13 is the "smaller half" of 27; $584 = 2 \times 292$)
	6×1168	(6 is the "smaller half" of 13)
*	3×2336	
*	1×4672	

Now add the numbers in the second column which are opposite an odd number in the first column (these are marked *):

$$292 + 584 + 2336 + 4672 = 7884.$$

This is the required result: $108 \times 73 = 7884$. Incredibly, the answer is correct. Wouldn't you think that throwing away all those halves would lead to chaos?

Here is an explanation of the example given above:

	$108 = 54 \times 2$	$108 \times 73 = 54 \times 2 \times 73 = 54 \times 146$
	$54 = 27 \times 2$	$108 \times 73 = 27 \times 292$
*	$27 = 26 + 1$	$108 \times 73 = 26 \times 292 + 1 \times 292$
*	$26 = 13 \times 2$	$108 \times 73 = 13 \times 584 + 292$
*	$13 = 12 + 1$	$108 \times 73 = 12 \times 584 + 584 + 292$
	$12 = 6 \times 2$	$108 \times 73 = 6 \times 1168 + 584 + 292$
	$6 = 3 \times 2$	$108 \times 73 = 3 \times 2336 + 584 + 292$
*	$3 = 2 + 1$	$108 \times 73 = 2 \times 2336 + 2336 + 584 + 292$
*	$2 = 1 \times 2$	$108 \times 73 = 4672 + 2336 + 584 + 292$

The left-hand equations provide the principle on which the method rests. The statements there can be combined to yield

$$\begin{aligned}
108 &= 54 \times 2 \\
&= 27 \times 2^2 \\
&= (26 + 1) \times 2^2 \\
&= 26 \times 2^2 + 2^2 \\
&= 13 \times 2^3 + 2^2 \\
&= (12 + 1) \times 2^3 + 2^2 \\
&= 12 \times 2^3 + 2^3 + 2^2 \\
&= 6 \times 2^4 + 2^3 + 2^2 \\
&= 3 \times 2^5 + 2^3 + 2^2
\end{aligned}$$

$$= (2+1)\times 2^5 + 2^3 + 2^2$$
$$= 2^6 + 2^5 + 2^3 + 2^2,$$

that is, the working serves to express 108 in binary notation as a sum of powers of 2. The doubling procedure is just a convenient way of calculating

$$2^2\times 73 = 292,\ \ 2^3\times 73 = 584,\ \ 2^5\times 73 = 2336 \ \text{and}\ \ 2^6\times 73 = 4672$$

Finally adding these numbers gives the product $108\times 73 = 7884$.

Stuart Robb,
Year 8,
Newington College.

Editor's comments.

Do you find the explanation convincing? Try out Russian multiplication on 4968×1234, say, and see how efficient it is.

You may like to investigate some other topics from antiquity. The Egyptians had difficulties with fractions: their notation only allowed them to write down fractions with numerator 1, that is they were forced to write ¼ + ½ instead of ¾, and so on. Can every fraction be written as a sum of such Egyptian fractions with numerator 1? If so, how?

Another topic for speculation arises from Roman numerals. How much did Galacticus Caesar pay for a gang of CVIII slaves at LXXIII dollars apiece? Can you work it out in Roman numerals?

ELEVENIFICATION

Dear Sir,

Under the heading of "Elevenification" in Parabola, Volume 15, Number 1, the following question was raised: If a number is added to its reversal and the same process done to the resulting sum repeatedly, does a multiple of 11 always result? (For example, $167+761 = 928$, $928+829 = 1757$, $1757+7571 = 9328 = 11\times 848$). I will show that this always works.

Suppose we start with the number

$$a.10^n + b.10^{n-1} + c.10^{n-2} + \ldots + x.10^2 + y.10 + z.$$

The sum of the number and its reversal is

$$(a+z)10^n + (b+y)10^{n-1} + (c+x)10^{n-2} + \ldots + (c+x)10^2 + (b+y)10 + (a+z),$$

and this is divisible by 11 if the alternating sum

$$(a+z) - (b+y) + (c+x) - \ldots$$

is divisble by 11. If n is odd, this is clearly so since the alternating sum is

$$(a+z) - (b+y) + (c+x) - \ldots - (c+x) + (b+y) - (a+z) = 0.$$

If n is even, this argument fails, but by continuing the process of reversal and addition, we will eventually obtain a number for which n is odd and then one further reversal and addition gives a multiple of 11.

Ross Baldick,
Year 12, Chatswood High School.

THE NATIONAL MATHEMATICS SUMMER SCHOOL

Dear Sir,

I am writing to share my experiences at the 1979 National Mathematics Summer School which was held at Bruce Hall, ANU, from the 8th to the 20th of January earlier this year. This was the eleventh such Summer School, sponsored by the Australian National University, the Australian Association of Mathematics Teachers, and other organisations. This annual summer school provides an excellent opportunity for senior students (about to enter Year 12, plus some post-HSC people) to meet together for a fortnight in a curriculum designed to promote skills in mathematics and encourage "scientific" thinking. I am sure that all of the seventy-odd students who attended this year's summer school enjoyed the opportunity of mingling with people who have similar interests and abilities.

In New South Wales, invitations to the Summer School (this State contributes twenty of the post-year 11 participants) are substantially determined by performance in the NSW Mathematics Olympiad examination, which is open to Year 11 students and held in August.

This year's summer school was, I gather, fairly unique, for the weather was unbearably *hot!* For the first week we had heat-wave conditions, which made concentration difficult; the second week was not so bad, but still hot. The weather, of course, promoted such spare time activities as swimming, and the canteen was kept busy selling cold drinks.

A typical day in the life of a Summer School student might be as follows: wake up at 7:00, breakfast at 8:00, an hour's lecture at 9:00, tutorials at 10:00, lunch at 12:30, a lecture at 1:30, with the hours between 3:00 and 6:00 free for recreation or study. Students were encouraged to get a balance between study and recreation. After dinner at 6:00, there might be an evening lecture, free time, or an organised outing. Some of the excursions organised by the School included visits to the Olympic Pool, tours of Canberra, a visit to the Academy of Science for a lecture, and an excursion to Mount Stromlo observatory. Sunday January 14 was left vacant for optional activities such as church services, bike riding, boating, golf (or even working).

The main lectures were on number theory, and were held every morning except Sunday for the duration of the school. Our Lecturer was Professor Arnold Ross from Ohio State University. He is an excellent lecturer in his subject and his mannerisms are famous at the Summer School. We commenced our studies by considering the properties of integers, developing a list of properties which defined the set of integers. From here we adapted to Gaussian integers ("integers in two dimensions") and polynomials in both the sets of integers and Gaussian integers. The properties of integers are fundamental to a mathematical approach to the study of numbers, since our understanding of the classification of numbers is based on an intuitive grasp of the term "integer". The ground covered in the lectures was revised each day in our tutorial periods, and exercises were done every day.

Dr John Mack from Sydney University lectured each afternoon on Regular Patterns and Geometry. He commenced his lectures by talking about maps, their definition, usage, and the various properties of different types of projections. After consideration of this field, we discussed transformations in one dimension and two-dimensional Euclidean plane geometry. We later spent a few enjoyable periods considering patterns in the plane, and even non-Euclidean geometries (especially the hyperbolic geometry).

Dr Paul Scott from Adelaide University lectured in the evenings for our first week at Canberra, and he dealt with the topic of Lattice Points. He dealt with problems involved in square lattices and rhombic lattices, and encouraged us to try and find a problem in these areas which had not been previously researched. He also kept us enthralled with his stories of the thesis he wrote for his doctorate: methods of finding maxima and minima for quadratic functions in seven variables, or how best to pack seven-dimensional spheres.

I hope I have been able to give you a good idea of the 1979 ANU/AAMT Summer School. If my attitude has been slightly frivolous in parts, it merely reflects the great enjoyment had by participants in the School.

P. Stott,
Year 12,
Newington College.

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PUZZLE FORLORN

What is the most efficient way to mow a rectangular lawn? Let us agree to ignore complications such as potted geraniums, dogs and garden gnomes and consider the problem of mowing an ideal lawn as follows. We have to do work in walking along behind the mower and we can assume this is proportional to the distance walked. We also have to do work in turning the mower round and we can assume this is proportional to the angle through which we turn. According to a rich friend who owns a power mower, the second sort of work is more enervating than the first. Below are three suggested routes for covering the lawn. Can you work out which of these requires the least work? Can you find a solution which is better than any of these? Do you think our assumptions are realistic?