COINCIDENCE RESOLVED

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In Volume 15, Number 1, we left you the following problem. (See "Triangulation" by D. McGrath.) Take a scalene triangle ABC and produce the sides as shown in Figure 1 so that AS = AT = ½(AC − AB), BV = BW = ½(BC − BA) and CY = CZ = ½(CA − CB). Let P, Q and R be the mid-points of the sides BC, AC and AB. You will easily check that this produces the same figure as the construction in the earlier article. The facts to be explained are as follows:

(1a) The points S, T and P are collinear,
(1b) The points W, V and Q are collinear,
(1c) The points Z, Y and R are collinear, and
(2) The lines SP, WQ and ZR are concurrent.

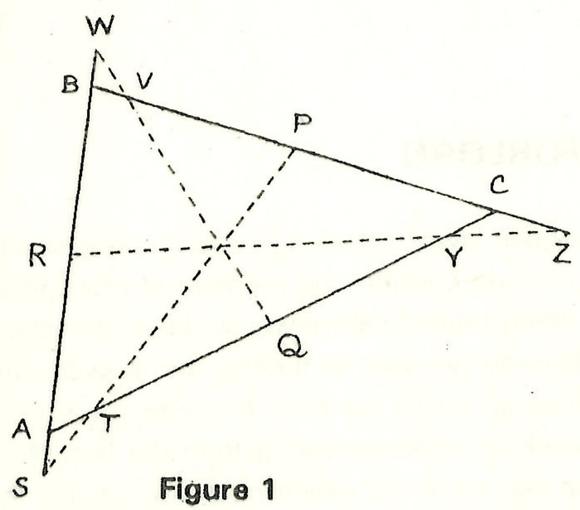

Figure 1

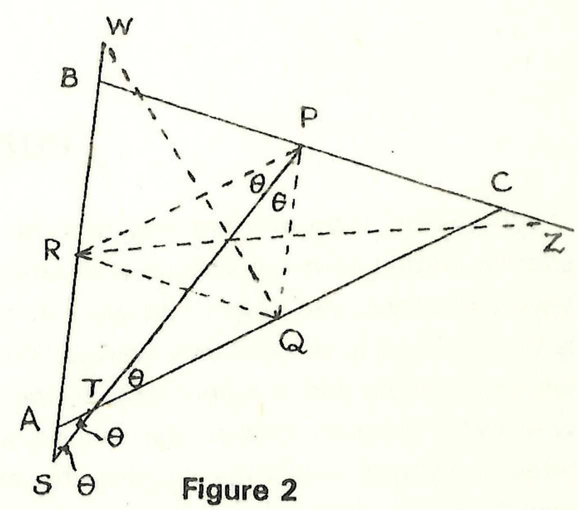

Figure 2

Now turn to Figure 2, which is just Figure 1 with certain irrelevant lines removed. Since AS = AT, the angles AST and ATS are equal. Since P and Q are midpoints of their respective sides of triangle ABC, the lines PQ and AB are parallel and the angles AST and TPQ are equal. Moreover, PQ = ½AB and TQ = AQ − AT = ½AC − ½(AC − AB) = ½AB, so the triangle PQT is isosceles and the angles TPQ and PTQ are equal. Putting all this together, we have shown that the angles ATS and PTQ are equal, so the segments ST and TP must form a straight line. This gives (1a); the same method also yields (1b) and (1c).

Finally consider triangle PQR. Since P and R are midpoints, the line PR is parallel to AC and so the alternate angles RPT and PTQ are equal. From what we have already shown, this means that SP bisects the angle QPR. By symmetry, WQ bisects the angle PQR and ZR bisects the angle QRP. So the three lines SP, WQ and ZR meet at the incentre of the triangle PQR. This proves (2).