PROBLEM SECTION

You are warmly invited to submit solutions to one or more of the problems that follow. Please begin each problem on a new page and make sure that each answer bears your name, year and school. Solutions, together with the names of successful solvers, will be published in the issue after next.

Happy puzzling.

429. Let a be a positive integer. Prove that the fraction $(a^3+2a)/(a^4+3a^2+1)$ is in its lowest terms.

430. Let p be a prime greater than 3. Show that p^2 is one more than a multiple of 12.

431. Adjoin to the digits 632 . . . three more digits so that the resulting six digit number is divisible by each of 7, 8 and 9.

432. Find all three digit numbers which are equal to the sum of the factorials of their digits.

433. Which is larger, 100^{300} or $300!$?

434. Show that if N is taken sufficiently large, the sum $1+2^{-1}+3^{-1}+4^{-1}+\ldots+N^{-1}$ is larger than 100. (See the articles on "Bricks that almost topple" in Parabola, Volume 14, Number 3 and Volume 15, Number 2.)

435. Show how to cut a square piece of paper into acute-angled triangles.

436. Find all solutions in non-negative integers x,y of the equation $3.2^x+1 = y^2$.

437. Two hundred points are distributed in space so that no three are collinear and no four are coplanar. Prove that it is possible to draw 10,000 line segments joining them without completing a single triangle.

438. Prove that every positive integer can be written as the sum of distinct Fibonacci numbers. (The Fibonacci numbers F_n are defined by $F_1 = 1$, $F_2 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$).

439. Given any two people, we may classify them as friends, enemies, or strangers. Prove that at a gathering of seventeen people, there must be either (i) three mutual friends, (ii) three mutual enemies, or (iii) three mutual strangers.

440. Let Π be a polygon and let P be any point inside Π. If every line segment joining P to any other point inside or on Π lies completely in Π, we say that Π is visible from P. For example, in Figure 1, the polygon is visible from A but not from B because the line segment BQ passes outside the polygon. Prove that the set of all points from which Π is visible is a convex set. (A set S in the plane is convex if the line segment joining any two points A and B of S lies completely in S — see Figure 2.)

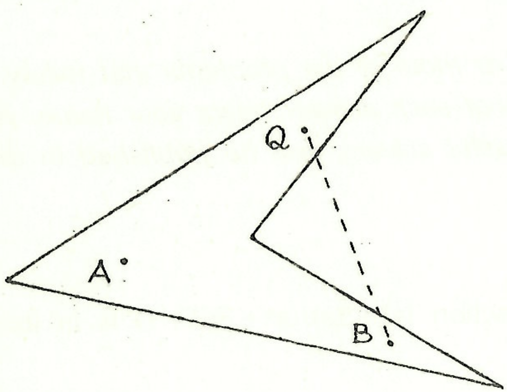

Figure 1

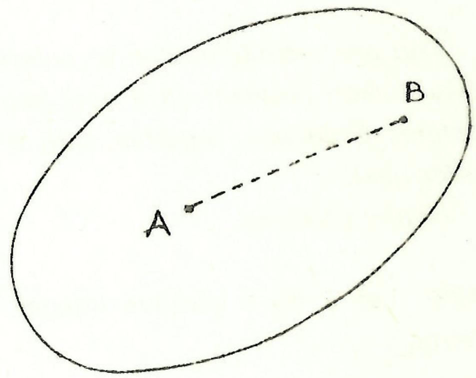

Figure 2

SOLUTIONS TO PROBLEMS FROM VOLUME 15, NUMBER 1

405. If k and N are positive integers with $k > 1$, show that it is possible to find N consecutive odd integers whose sum is N^k.

Solution:

The average size of the N summands must be equal to their sum divided by N, that is N^{k-1}. Hence if N is odd, we take the N consecutive odd integers

$$N^{k-1} - N + 1,\ N^{k-1} - N + 3,\ \ldots,\ N^{k-1} - 2,\ N^{k-1},\ N^{k-1} + 2,\ \ldots,\ N^{k-1} + N - 1$$

with the middle one equal to N^{k-1}. If N is even, we take the N consecutive odd integers

$$N^{k-1} - N + 1,\ N^{k-1} - N + 3,\ \ldots,\ N^{k-1} - 1,\ N^{k-1} + 1,\ \ldots,\ N^{k-1} + N - 1$$

with half the numbers lying on either side of N^{k-1}.

Correct solutions were received from Ross Baldick (Chatswood High School), D.S. McGrath (The King's School), Richard Wilson (The King's School) and Otis Wright (Davidson High School).