439. Given any two people, we may classify them as friends, enemies, or strangers. Prove that at a gathering of seventeen people, there must be either (i) three mutual friends, (ii) three mutual enemies, or (iii) three mutual strangers.

440. Let Π be a polygon and let P be any point inside Π. If every line segment joining P to any other point inside or on Π lies completely in Π, we say that Π is visible from P. For example, in Figure 1, the polygon is visible from A but not from B because the line segment BQ passes outside the polygon. Prove that the set of all points from which Π is visible is a convex set. (A set S in the plane is convex if the line segment joining any two points A and B of S lies completely in S — see Figure 2.)

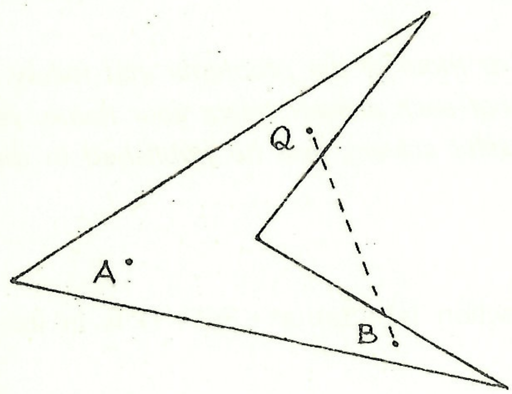


Figure 1

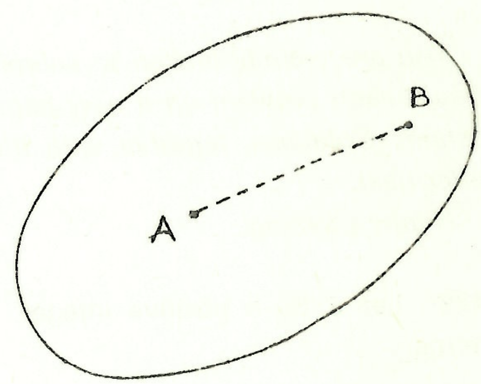


Figure 2

SOLUTIONS TO PROBLEMS FROM VOLUME 15, NUMBER 1

405. If k and N are positive integers with $k > 1$, show that it is possible to find N consecutive odd integers whose sum is N^k.

Solution:

The average size of the N summands must be equal to their sum divided by N, that is N^{k-1}. Hence if N is odd, we take the N consecutive odd integers

$$N^{k-1} - N + 1,\ N^{k-1} - N + 3,\ \ldots,\ N^{k-1} - 2,\ N^{k-1},\ N^{k-1} + 2,\ \ldots,\ N^{k-1} + N - 1$$

with the middle one equal to N^{k-1}. If N is even, we take the N consecutive odd integers

$$N^{k-1} - N + 1,\ N^{k-1} - N + 3,\ \ldots,\ N^{k-1} - 1,\ N^{k-1} + 1,\ \ldots,\ N^{k-1} + N - 1$$

with half the numbers lying on either side of N^{k-1}.

Correct solutions were received from Ross Baldick (Chatswood High School), D.S. McGrath (The King's School), Richard Wilson (The King's School) and Otis Wright (Davidson High School).

406. Given n beads numbered 1, 2, 3, . . ., n, show how you can make a single-strand closed necklace from them with the property that the numbers on adjacent beads always differ by either 1 or 2.

Solution from D.S. McGrath (The King's School), Richard Wilson (The King's School) and Otis Wright (Davidson High School):

If n is odd, arrange the beads thus:

$$1, 3, 5, 7, 9, \ldots, n-2, n, n-1, n-3, \ldots, 4, 2, 1, 3, \ldots$$

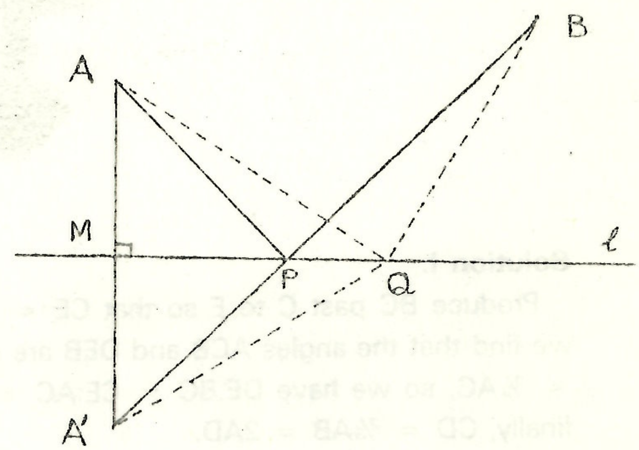
If n is even, arrange the beads thus:

$$2, 4, 6, 8, 10, \ldots, n-2, n, n-1, n-3, \ldots, 5, 3, 1, 2, 4, \ldots$$

Correct solutions were also received from Ross Baldick (Chatswood High School), Stephen Pereira (St. Patrick's College, Goulburn), Jennifer Taylor (Woy Woy High School) and Surinder Wadhwa (Ashfield Boys High School).

407. Let ℓ be a given line and let A and B be two points on the same side of ℓ, as shown in the figure. Find the point P on ℓ with the property that the sum of the distances AP and PB is as small as possible. Prove that your answer is correct.

Solution from D.S. McGrath (The King's School):



Let AA′ be perpendicular to ℓ with AM = A′M. For any point Q on ℓ, the triangles AMQ and A′MQ are congruent and so AQ = A′Q and AQ+QB = A′Q+QB. Now, if A′PB is straight, we have $A'P+PB \leqslant A'Q+QB$. Hence AP+PB is least for this position of P. Note that this is the path of the light ray from A to B reflected on the line ℓ.

Correct solutions were also received from Richard Wilson (The King's School) and Otis Wright (Davidson High School).

408. The number 3 can be expressed as the sum of one or more positive integers in 4 ways: 3, 2+1, 1+2 and 1+1+1. Note that the ordering of the summands is significant; 1+2 is counted as well as 2+1. Find a formula for the number of ways in which an arbitrary positive integer n can be so expressed as a sum of positive integers. Prove that your answer is correct.

Solution from Ross Baldick (Chatswood High School) and D.S. McGrath (The King's School):

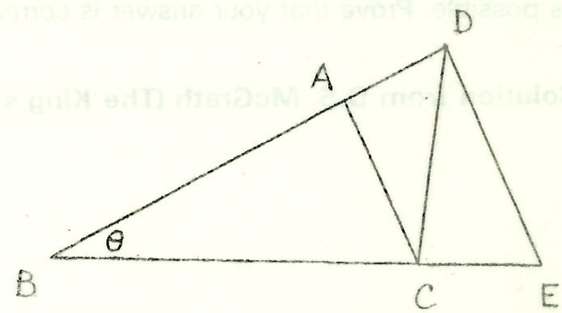
Consider the partitions of 3:

$$1|1|1 = 1+1+1$$
$$1,1|1 = 2+1$$
$$1|1,1 = 1+2$$
$$1,1,1 = 3$$

In the representations on the left, the "|" marks the end of a summand in the partition and the "," separates 1's in the same summand. For a partition of n, we have n 1's and $n-1$ spaces between them. Each ordered partition of n corresponds to an assignment of either "|" or "," into these $n-1$ spaces. There are 2^{n-1} ways of assigning these symbols, so there are 2^{n-1} different ordered partitions of n.

The correct formula, 2^{n-1}, was also supplied by Danny O'Keefe (The Scots School, Bathurst), Jennifer Taylor (Woy Woy High School), Surinder Wadhwa (Ashfield Boys High School) and Richard Wilson (The King's School).

409. In a triangle ABC, BC = 2AC. Produce BA past A to D so that AD = ⅓AB. Prove that CD = 2AD.



Solution I:

Produce BC past C to E so that CE = ⅓BC. Then the triangles ABC and DBE are similar and we find that the angles ACB and DEB are equal and that DE = (4/3)AC. From the construction, CE = ⅔AC, so we have DE:BC = CE:AC = 2:3. Thus the triangles CED and ACB are similar and, finally, CD = ⅔AB = 2AD.

Solution II from Ross Baldick (Chatswood High School), Richard Wilson (The King's School) and Otis Wright (Davidson High School):

Applying the cosine rule in triangle BCD gives

$$CD^2 = BD^2 + BC^2 - 2BD.BC \cos\theta = 16\ AD^2 + 4AC^2 - 16\ AD.AC \cos\theta.$$

Applying the cosine rule again in triangle ABC gives

$$\cos\theta = (AB^2+BC^2-AC^2)/\ 2AB.BC = (9AD^2+4AC^2-AC^2)/12\ AD.AC$$

If this is substituted in the equation for CD^2, everything magically simplifies to give CD = 2AD, as required.

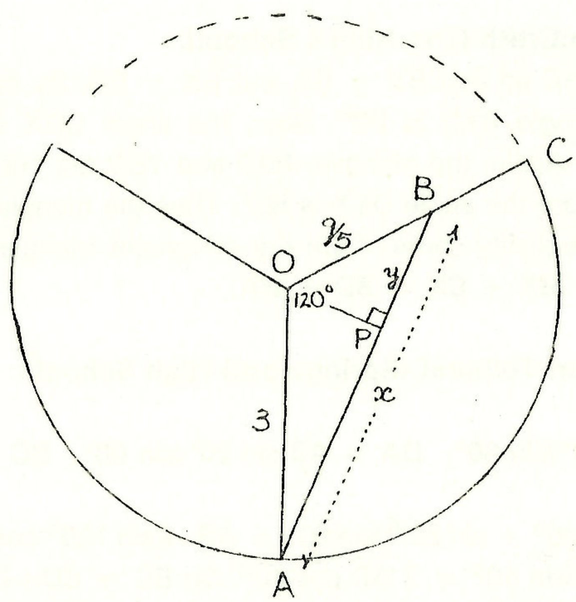
Another good solution was received from Surinder Wadhwa (Ashfield Boys High School).

410. Mount Zircon is shaped like a perfect cone whose base is a circle of radius 2 miles, and the straight line paths up to the top are all 3 miles long. From a point A at the southernmost point of the base, a path leads to B, a point on the northern slope and 2/5 of the way to the top. If AB is the shortest path on the mountainside joining A to B, find

(i) the length of the whole path AB, and

(ii) the length of the path between P and B, where P is a point on the path at which it is horizontal.

Solution from Ross Baldick (Chatswood High School) and Stephen Pereira (St. Patrick's College, Goulburn):



The diagram shows the surface of the cone flattened out into a plane figure. The distance from the summit, O, to A, the southernmost point of the base, is 3 miles and OB is 3/5 of the northern slope OC, a distance of 9/5 miles. The length of the arc AC is half the perimeter of the base, that is $\frac{1}{2}.2\pi.2 = 2\pi$ miles. Since this is one third of the circumference of a circle of radius 3 miles, the angle AOB is 120°. The horizontal point P on the path is the point P at which the line PO straight up the mountain is perpendicular to the path. So the problem reduces to solving the triangle OAB for the distances x and y. By the cosine rule,

$$x^2 = AB^2 = OA^2 + OB^2 - 2OA.OB \cos 120°,$$

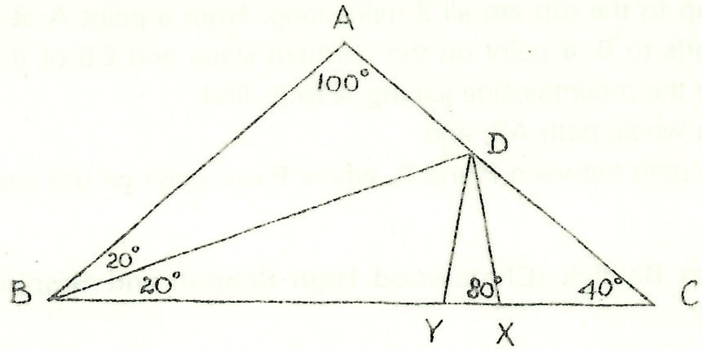
which gives $x = 21/5$ after simplification. Next

$$y = OB \cos\theta = OB\,(OB^2 + AB^2 - OA^2)/2OB.AB = 99/70$$

Thus AB = 21/5 miles and PB = 99/70 miles.

Correct solutions were also received from D.S. McGrath (The King's School) and Otis Wright (Davidson High School).

411. The angles ABD, DBC and BCD in the figure are 20°, 20° and 40° respectively. Prove that BC = BD + DA.



Solution I from D.S. McGrath (The King's School):

Locate points X,Y on BC so that BY = BA and BX = BD. By construction, the triangle BDX is isosceles, so the base angle BXD is 80°. Now, the angle CDX is 40°, so the triangle CDX is isosceles and CX = DX. Again, the triangles ABD and YBD are congruent, so angle BYD = angle BAD = 100° and therefore the angle DYX is 80°. Thus the triangle DXY is also isosceles and DX = DY = DA. (The last equality comes from the congruent triangles.) Altogether this gives CX = DA. Consequently, BC = BX + CX = BD + DA.

Solution II from Stephen Tolhurst (Springwood High School):

By the sine rule,

$$BD = AB \sin 100°/\sin 60°, \quad DA = AB \sin 20°/\sin 60°, \quad BC = AB \sin 100°/\sin 40°.$$

Now

$$BD + DA = AB\,.\,(\sin 100° + \sin 20°)/\sin 60° = AB\,.\,2 \sin 120° \cos 40°/\sin 60° = 2\,AB \cos 40°,$$

while $BC = AB\,.\,\sin 80°/\sin 40° = 2\,AB \cos 40°$. So BC = BD + DA as required.

A partial solution was received from Surinder Wadhwa (Ashfield Boys High School).

412. Consider a convex polygon with n vertices, and suppose that no three of its diagonals meet at the same point inside the polygon. Determine

(i) the total number of line segments into which the diagonals are divided by their points of intersection, and

(ii) the total number of compartments into which the figure is divided by all its diagonals.

Solution:

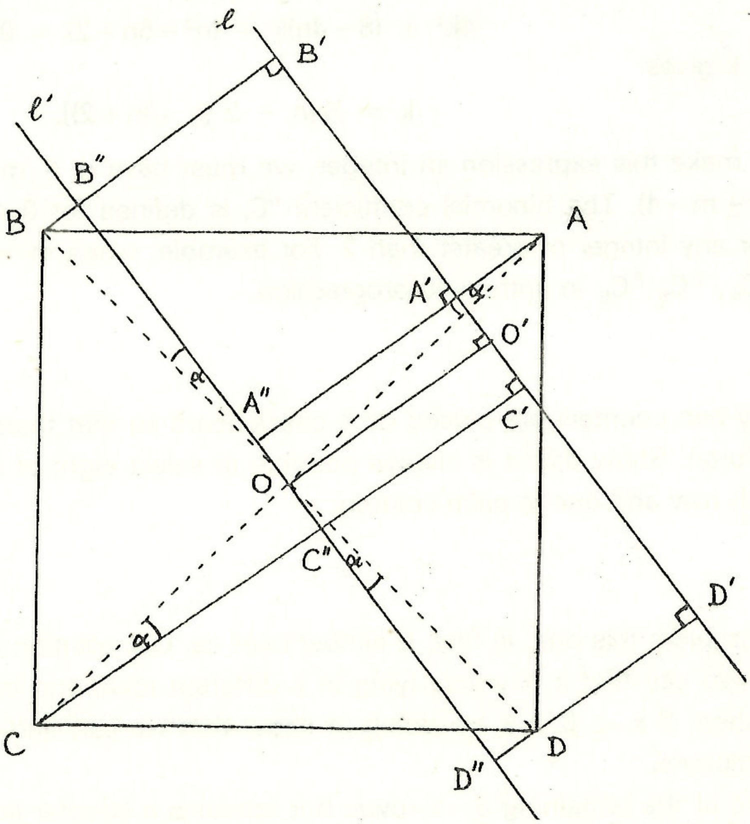
(i) The total number of points of intersection of the diagonals inside the polygon is $^nC_4$ since any selection of 4 of the vertices determines two diagonals having just one point of intersection inside the polygon, and conversely every such point of intersection arises in this way. Now imagine the diagonals drawn in succession. Each additional intersection point divides each of the two intersecting segments into two parts and hence increases by two the total number of segments into which the diagonals are divided. Hence the total number of these segments is equal to

$$\text{number of diagonals} + \text{twice number of points of intersection}$$
$$= \tfrac{1}{2}n(n-3) + 2.{}^nC_4 = n(n-3)(n^2-3n+8)/12.$$

(ii) Let F be the total number of compartments, V the total number of vertices (that is the vertices of the polygon and the points of intersection of the diagonals inside the polygon) and E the total number of edges (that is edges of the polygon and segments into which the diagonals are divided). Then Euler's formula gives $F - E + V = 1$. Hence

$$F = E - V + 1 = (n + \tfrac{1}{2}n(n-3) + 2.{}^{n}C_4) - (n + {}^{n}C_4) + 1 = (n-1)(n-2)(n^2-3n+12)/24.$$

413. In the accompanying figure, O is the centre of the square ABCD and ℓ is a given line. The points O′, A′, B′, C′ and D′ are the feet of the perpendiculars dropped from O, A, B, C and D to the line ℓ. If AA′.CC′ = BB′.DD′ and AB = 2 inches, find OO′. Prove that your answer is correct.



Solution:

Draw a line ℓ' through O and parallel to ℓ. Let A″, B″, C″ and D″ be the feet of the respective perpendiculars from A, B, C, and D onto ℓ.

Note that angle OAA″ = angle BOB″ = angle DOD″ = OCC″ = α, say.

Now OA = $\sqrt{2}$, so we have

$$AA'' = OA\cos\alpha = \sqrt{2}\cos\alpha,\quad BB'' = \sqrt{2}\sin\alpha,\quad CC'' = \sqrt{2}\cos\alpha,\quad DD'' = \sqrt{2}\sin\alpha.$$

If OO′ = x, then

$$AA' = \sqrt{2}\cos\alpha - x,\quad CC' = \sqrt{2}\cos\alpha + x,\quad BB' = \sqrt{2}\sin\alpha + x,\quad DD' = x - \sqrt{2}\sin\alpha.$$

Substituting in AA′.CC′ = BB′.DD′ gives

$$(\sqrt{2}\cos\alpha - x)(\sqrt{2}\cos\alpha + x) = (x + \sqrt{2}\sin\alpha)(x - \sqrt{2}\sin\alpha),$$

that is $2\cos^2\alpha - x^2 = x^2 - 2\sin^2\alpha$. So $2x^2 = 2$ and x = 1 inch.

414. Find all positive integers n and k such that the three binomial coefficients ${}^nC_k$, ${}^nC_{k+1}$ and ${}^nC_{k+2}$ are in arithmetic progression.

Solution from Stephen Pereira (St. Patrick's College, Goulburn):

If ${}^nC_k$, ${}^nC_{k+1}$ and ${}^nC_{k+2}$ are in arithmetic progression, then $2\,{}^nC_{k+1} = {}^nC_k + {}^nC_{k+2}$. Using ${}^nC_r = n(n-1)(n-2)\ldots(n-r+1)/r!$, we obtain after some simplification

$$2(n-k)(k+2) = (k+1)(k+2) + (n-k)(n-k-1),$$

whence

$$4k^2 + (8-4n)k + (n^2-5n+2) = 0.$$

Solving for k gives

$$k = \tfrac{1}{2}[n - 2 \pm \sqrt{(n+2)}].$$

In order to make this expression an integer, we must have $n = m^2-2$ for some integer m, giving $k = \tfrac{1}{2}(m^2 \pm m - 4)$. The binomial coefficient ${}^nC_r$ is defined for $0 \leqslant r \leqslant n$, so the above solutions are valid for any integer m greater than 2. For example, when $m = 3$, we get the triples ${}^7C_1$, ${}^7C_2$, ${}^7C_3$ and ${}^7C_4$, ${}^7C_5$, ${}^7C_6$ in arithmetic progression.

415. Thirty-two counters are placed on a chess-board so that there are four in every row and four in every column. Show that it is always possible to select eight of them so that there is one of the eight in each row and one in each column.

Solution:

A tricky problem this one; in fact, it almost beat us. Our solution is, we believe, suitably devious.

Suppose we can find k counters lying in k different rows and in k different columns. If $k = 8$, we are finished. If $k < 8$, we are going to show that we can find $k+1$ counters in $k+1$ different rows and columns.

Select one of the remaining $8-k$ rows. If it contains a counter in one of the remaining $8-k$ columns, we are finished since this counter qualifies as the desired $(k+1)$-th counter.

So now suppose no counter in any of the remaining $8-k$ rows is in any of the remaining $8-k$ columns. Select one of the remaining $8-k$ rows, R say, and one of the remaining $8-k$ columns, C say. Each of the four counters in R is in the same column as one of the k special counters already chosen. Likewise, four of these k counters are in the same row as a counter in C. But k is less than $4+4$. So there must be one of the k special counters, X, which is in the same column as a counter Y in row R and in the same row as a counter Z in column C. If we now replace the counter X by the two counters Y and Z, we have $k+1$ counters in $k+1$ different rows and columns.

416. Let S be a convex area which is symmetric about the point O. Show that the area of any triangle drawn in S is less than or equal to half the area of S.
(Definition: A set is symmetric about the point O if whenever a point A is in the set, so is the point B which lies at the same distance from O as A on the line AO produced.)

Solution:

Let ABC be any triangle in S. Reflect A, B and C in O to get the points A′, B′ and C′. If O lies outside the triangle ABC (Figure 3), then the congruent triangle A′B′C′ lies in S and does not overlap triangle ABC. (Note that corresponding sides of the two triangles are parallel.) Hence the area of S is greater than the sum of the areas of triangles ABC and A′B′C′, which is twice the area of triangle ABC.

If O lies inside the triangle ABC (Figures 1 and 2), then the hexagon AC′BA′CB′ lies inside S. We are finished when we show that the area of this hexagon is twice the area of triangle ABC. The following six pairs of triangles have equal areas: OAC′ and OAC, OBC′ and OBC, OBA′ and OBA, OCA′ and OCA, OCB′ and OCB, OAB′ and OAB. (In each case, the triangles in each pair have equal bases and the same perpendicular height.) Adding the areas of these triangles shows that the area of the hexagon AC′BA′CB′ is twice the area of the triangle ABC, as required.

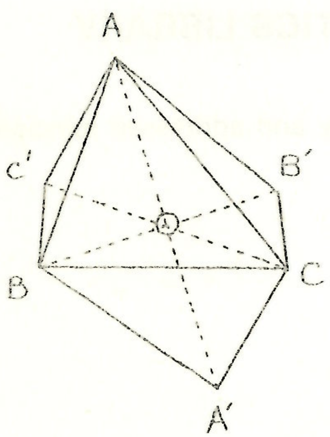


Figure 1

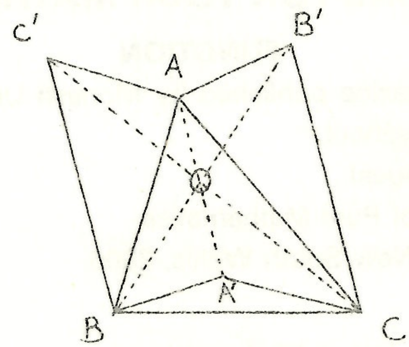


Figure 2

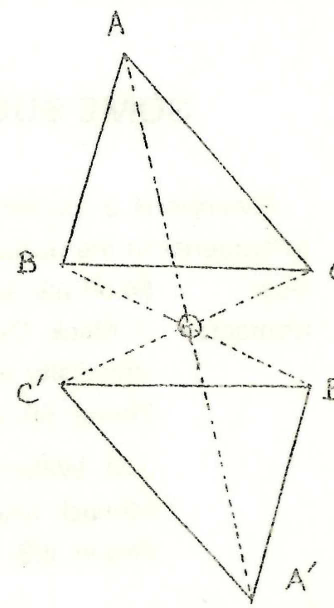
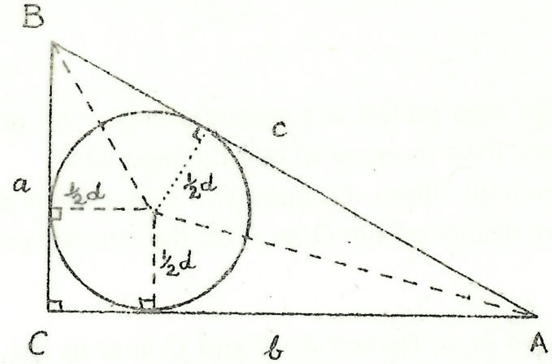


Figure 3

☆ ☆ ☆

SOLVERS OF EARLIER PROBLEMS

The following contributions were received too late for acknowledgement in the last issue: Michael Hall (Hawker College): solutions to problems 393, 394, 397, 399, 400 and partial solutions to 396, 398 and 402. Michael gives the following neat solution for problem 400.



We want to show that the diameter d of the incircle of the right-angled triangle in the figure satisfies $d = a+b-c$. The area of the right-angled triangle is $\frac{1}{2}ab$; alternatively, by breaking the figure up into a square and three triangles as shown, this area is

$$(\tfrac{1}{2}d)^2 + \tfrac{1}{2}(\tfrac{1}{2}d)(a-\tfrac{1}{2}d) + \tfrac{1}{2}(\tfrac{1}{2}d)(b-\tfrac{1}{2}d) + \tfrac{1}{2}(\tfrac{1}{2}d)c = \tfrac{1}{4}d(a+b+c).$$

Equating these two expressions for the area gives

$$d = 2ab/(a+b+c).$$

Now, by Pythagoras' theorem, $c^2 = a^2 + b^2 = (a+b)^2 - 2ab$, that is

$$2ab = (a+b)^2 - c^2 = (a+b+c)(a+b-c).$$

So we get $d = a+b-c$.

☆ ☆ ☆

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Function is a mathematics magazine published by Monash University and addressed principally to students in the upper forms of school.

Cost: $4.00 per annum (5 issues).

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