LETTERS TO THE EDITOR

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1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 1 1 5 1
1 6 6 2 6 6 1
1 7 3 8 8 3 7 1
1 8 1 2 7 2 1 8 1

Dear Sir,

In Parabola, Volume 15, Number 2, Paul Rider described a modified Pascal's triangle, shown above. To get each new entry in the triangle, add the two numbers above and immediately to the left and right as usual, but if this gives a two-digit number, add the digits together and write this answer into the triangle. I noticed that a row of this triangle is eleven times the previous row if and only if the additions performed to produce the row give only single digit sums. To see why, consider the formation of the next row

1 9 9 3 9 9 3 9 9 1.

All we have done is to add the previous row to ten times itself, thus multiplying the previous row by eleven. At the next step, we get

1 10 18 12 12 18 12 12 18 10 1.

If we carry each tens digit over to the left as shown we get

2 1 9 3 3 9 3 3 9 0 1

which is eleven times the previous row, just as before. However, if we add the digits of the two-digit numbers together, giving

1 1 9 3 3 9 3 3 9 1 1,

we get a different result whenever two-digit sums occur as they do here. As we go further down the table, it becomes more and more unlikely that all the sums are single-digit numbers, so we do not expect to find too many more pairs of rows, one eleven times the other, in the modified Pascal's triangle.

Scott Driver,
Form 6,
The King's School.

UP, UP AND AWAY

Dear Sir,

After reading "Bricks that almost topple over again" (Parabola, Volume 15, Number 2), I programmed the problem for the PDP 11/70 computer I was working on at the time. The program calculated the maximum overhang that can be obtained with a pile of n bricks by adding the terms of the series

$$1/2 + 1/4 + 1/6 + \ldots + 1/2n.$$

The computer has an accuracy of 15 significant figures, so is much more accurate and, of course, quicker than a pocket calculator. Even so, after an overnight run, the computer had only reached a 9-brick overhang!

Number of bricks	Maximum overhang
4	1.04
31	2.01
227	3.002
1 674	4.000 2
12 367	5.000 02
91 380	6.000 002
675 214	7.000 000 7
4 989 191	8.000 000 05
36 865 412	9.000 000 002

Andrew Johnston,
Year 10,
Ignatius Park College, Townsville.

Editor's comments:

There is a moral to this story: the significance of the numbers churned out by a computer requires careful thought both before and after the run. Consider the case of the 9-brick overhang. To get this, we add about 3×10^7 terms, each operation producing a potential error of about 10^{-14} in the sum. Of course, these are "random" errors, some positive, some negative. Thus, although the total error might be as large as $10^{-14} \times 3 \times 10^7 = 3 \times 10^{-7}$, it is probably much less. In any case, it pays to be suspicious, since we need to calculate the overhang to within about 10^{-8} to get the correct number of bricks. Let us compare the above results with those obtained by the method described in Volume 15, Number 2.

Number of bricks	Maximum overhang
4 989 191	8.000 000 05
36 865 412	9.000 000 005
272 400 598	10.000 000 000 5

As you can see, the computer has reached its limit in both time and accuracy. Incidentally, taking a common or garden standard brick measuring 8¾ inches long by 3 inches high, our pile of 272 400 598 bricks with an overhang of just 7 feet 3½ inches is 13,000 miles high

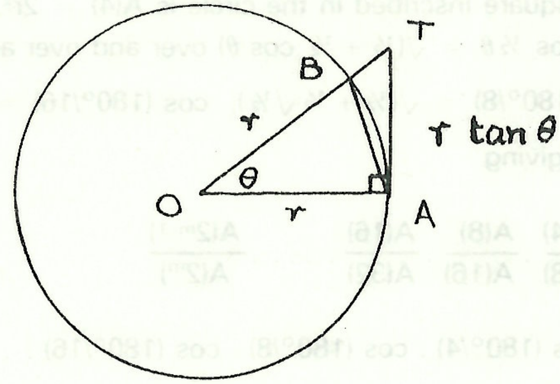
ANOTHER APPROACH TO PI

Dear Sir,

In a letter in Parabola, Volume 15, Number 2, Richard Wilson explained why the number n sin (180°/n) tends to π as the integer n grows larger and larger. This was shown by comparing the area of a circle with that of a polygon with n sides inscribed in the circle, or circumscribed about it. If we put $\theta = 180°/n$, the limit for π becomes

$$(180°/\theta) \sin \theta \to \pi \quad \text{as} \quad \theta \to 0.$$

(Note that the angle θ is measured in degrees.) Here is another way to derive this limit.



In the figure, we have a circle with centre O and radius r, say, and AT is a tangent to the circle at A.

Now

area of triangle OAB $\leqslant$ area of sector OAB $\leqslant$ area of triangle OAT,

that is

$$\tfrac{1}{2} r^2 \sin \theta \leqslant \pi r^2 \theta/360 \leqslant \tfrac{1}{2} r^2 \tan \theta.$$

If we divide all the terms by $\tfrac{1}{2} r^2 \sin \theta$, this becomes

$$1 \leqslant \pi\theta/180 \sin \theta \leqslant 1/\cos \theta.$$

As $\theta \to 0$, we have $\cos \theta \to 1$, so $\pi\theta/180 \sin \theta \to 1$ as well, that is $(180°/\theta) \sin \theta \to \pi$.

Ramses Youhana,
Year 10,
North Sydney Boys' High School.

Editor's comments.

In essence, this argument uses the same geometric idea as the one described by Richard Wilson. Let us compare the two conclusions

(1) $(180°/\theta) \sin \theta \to \pi$ as $\theta \to 0$, and (2) $n \sin (180°/n) \to \pi$ as $n \to \infty$.

In one sense, the first result is better because the second is restricted to the particular sequence of angles $\theta = 180°/n$. However, if we are interested in using the result to get an approximation for π, then we need the value of sin θ. Now it is cheating to use a table of sines or a calculator to do this; we might just as well use our tables or calculator to get π without all this fuss. Our strategy therefore is to start with some convenient angle, say $\theta = 180°/3 = 60°$ and use the double angle formulae to calculate $\sin\theta$ successively for $\theta = 180°/(2.3), 180°/(2^2.3), 180°/(2^3.3), \ldots$ until the angle is small enough to give a good value for π. This is why the second limit is historically important. This idea can also be used to obtain a fascinating formula for π, first discovered by Francois Vieta in 1579. Suppose we inscribe a regular n-sided polygon inside the circle of radius r and call its area A(n). If we take $\theta = 360°/n$ in the figure above, then our polygon consists of n copies of the triangle OAB, so $A(n) = \frac{1}{2}nr^2\sin(360°/n)$. Doubling the number of sides gives a regular 2n-sided polygon with area $A(2n) = nr^2\sin(180°/n)$, and so

$$A(n)/A(2n) = \tfrac{1}{2}\sin(360°/n)/\sin(180°/n) = \cos(180°/n). \qquad (3)$$

In particular, the area of a square inscribed in the circle is $A(4) = 2r^2$, so we can start with $n = 4$. Next, we use the formula $\cos\frac{1}{2}\theta = \sqrt{(\frac{1}{2} + \frac{1}{2}\cos\theta)}$ over and over again giving

$\cos(180°/4) = \sqrt{\tfrac{1}{2}}$, $\cos(180°/8) = \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})}$, $\cos(180°/16) = \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})})}, \ldots$

We also use (3) repeatedly giving

$$\frac{A(4)}{A(2^m)} = \frac{A(4)}{A(8)} \cdot \frac{A(8)}{A(16)} \cdot \frac{A(16)}{A(32)} \cdot \ldots \cdot \frac{A(2^{m-1})}{A(2^m)}$$

$$= \cos(180°/4) \,.\, \cos(180°/8) \,.\, \cos(180°/16) \,.\, \ldots \,.\, \cos(180°/2^{m-1})$$

$$= \sqrt{\tfrac{1}{2}} \,.\, \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})} \,.\, \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})})} \,.\, \ldots \,.\, \sqrt{(\ldots)} \qquad (4)$$

where the last factor contains $m - 2$ square-root signs. What happens when m tends to infinity? On the right of (4), we get an infinite product. On the left, the term $A(2^m)$ is the area of a regular polygon with 2^m sides which approaches the area of the circle in which it is inscribed, that is $A(2^m) \to \pi r^2$ as $m \to \infty$. Since $A(4) = 2r^2$, we get Vieta's formula

$$2/\pi = \sqrt{\tfrac{1}{2}} \,.\, \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})} \,.\, \sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}})})} \ldots$$

TO – OR NOT TO –

Dear Sir,

In various texts, $\tan(\cos^{-1}x)$ is given as $\pm x^{-1}\sqrt{(1 - x^2)}$. However, suppose we put $y = \cos^{-1}x$, so that y is the (unique) solution of the equation $\cos y = x$ with $0 \leqslant y \leqslant \pi$. Then $\sin y \geqslant 0$, so

$$\sin y = \sqrt{(1 - \cos^2 y)} = \sqrt{(1 - x^2)} \qquad (1)$$

and

$$\tan y = \sin y / \cos y = x^{-1}\sqrt{(1 - x^2)}, \qquad (2)$$

that is

$$\tan(\cos^{-1}x) = x^{-1}\sqrt{(1 - x^2)}.$$

Could you please resolve my problem: does $\tan(\cos^{-1}x)$ have, imply, or otherwise need a $\pm$ sign?

Kieran Lim,
Year 11,
St. Ignatius' College.

Editor's comments.

Believe it or not, text-books can be wrong. Kieran is quite correct in his conclusion that $\tan(\cos^{-1}x) = x^{-1}\sqrt{(1 - x^2)}$. In fact, observe that $\tan(\cos^{-1}x)$ has a uniquely defined value for each value of x with $-1 \leqslant x \leqslant 1$, because we have carefully specified the definition of $\cos^{-1}x$: $y = \cos^{-1}x$ means that y is the unique solution of the equation $\cos y = x$ with $0 \leqslant y \leqslant \pi$. If we explore this a bit, we can see why the books play safe by including a $\pm$ sign. For example, let us pretend that $y = \cos^{-1}x$ means that y is the solution of $\cos y = x$ in the range $\pi \leqslant y \leqslant 2\pi$. Then $\sin y \leqslant 0$, so

$$\sin y = -\sqrt{(1 - \cos^2 y)} = -\sqrt{(1 - x^2)},$$

and

$$\tan(\cos^{-1}x) = \tan y = -x^{-1}\sqrt{(1 - x^2)}.$$

So the sign of our answer does depend on the way $\cos^{-1}x$ is defined and the $\pm$ sign allows for all possibilities. However, it is worth repeating that with the usual convention, as in Kieran's letter, there is only one answer and no reason at all for the ambiguous sign.

ANOTHER SUMMER SCHOOL?

Dear Sir,

I read with interest P. Stott's letter on the National Mathematics Summer School and I would like to correct a few misconceptions. Firstly, one does not just "wake up" at the Summer School. One is woken up, either by half a dozen assorted alarm clocks set by all the other people to wake up nice and early to stand in the breakfast queue for 20 minutes, or one is shaken awake by a well-meaning colleague who wants to discuss something important like the likelihood of playing cards with the good-looking girls upstairs. P. Stott didn't seem to emphasize enough what I think is the major purpose of the School: to bring together talented students in a situation where they are sharing experiences (not just mathematical) with people of similar intelligence, who have rather varied perceptions of the world and mathematics. It is very enjoyable to experience such a meeting, especially away from home in a new environment similar, apparently, to life at university.

N. Bourbaki,
Year 12,
Chatswood High School.

Editor's comments.

Your Editor would like to hear from any participants in this year's Summer School wishing to share their experiences with other readers. Incidentally, your Editor has some reservations about the authenticity of the author of this letter; a possible French connection is being investigated.