MORE BINARY MAGIC

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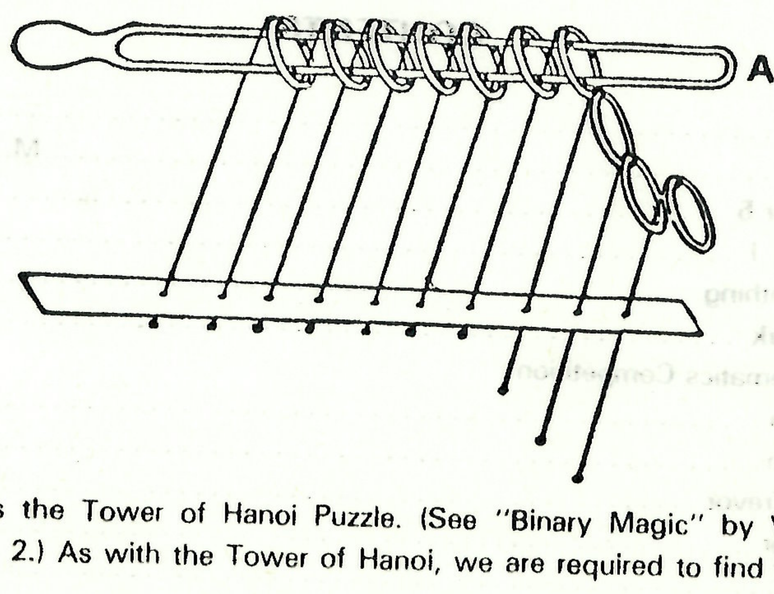
Chinese rings

Meleda, or Chinese rings, is a game of Chinese origin which, so the story goes, was invented by the soldier-hero Hung Ming (181-234 A.D.) who gave it to his wife when he went to war. In trying to solve the puzzle, she forgot to grieve for her husband. Early examples of the game have been found in England and Scandinavia. In fact, in Norway, the puzzle was even put to practical use as a rather effective luggage lock. As we will see, only a very determined or desperate thief would succeed in opening a trunk locked in this way, without resorting to bolt-cutters.

The game consists of a number of rings hung upon a bar in such a manner that the ring at the free end (marked A in the figure) can be taken off or put on at will. Any other ring can be put on or taken off only when the one next to it closer to the free end is on the bar and all others be tween that ring and the free end are off the bar. The order of the rings cannot be changed. It is quite simple to make your own meleda puzzle. Indeed, you will probably have to make one in order to see how it works. My own model is made of wood, using dowelling for the rods and wooden curtain rings for the rings which are attached to the base by brass chain. (Cardboard and string will do just as well for the rods and connections. A small number of rings is best for a start.) The puzzle is depicted in the figure below. Note that there is one slight exception to the rule of removal of the rings. This is that the first two rings may be taken off or put on together, whereas all other rings can be taken off or put on only one at a time. This is unimportant and, in the following discussion, it is assumed that only one ring is taken off or put on at a time.



Meleda resembles the Tower of Hanoi Puzzle. (See "Binary Magic" by V. Paul in Parabola, Volume 15, Number 2.) As with the Tower of Hanoi, we are required to find the shortest method

**This article comes from a reader in Canberra.*

of removing all the rings from the bar (by providing a list of the numbers of the rings moved at each step) and to determine the corresponding number of moves. Each move consists of putting on a ring or taking one off. The usual scheme is to number the rings from the free end and to commence with all rings on the bar. This problem was attacked unsuccessfully by the Italian mathematician Girolamo Cardano (c. 1550) and also by the English mathematician John Wallis (c. 1693).

Once the mechanics of the puzzle have been mastered, it is easy to find the moves required to take all the rings off the bar for 1, 2, 3 and 4 ring puzzles. This is shown in the following table.

Number of rings	Number of moves	Sequence of moves
1	1	1
2	2	2 1
3	5	1 3 1 2 1
4	10	2 1 4 1 2 1 3 1 2 1

The sequence of moves resembles the X Y X Z X Y X rule of the Tower of Hanoi. (See the article on "Binary Magic" cited earlier.) Further experimentation reveals just how closely related the two games are. The following table illustrates this connection.

Number of rings or discs	Tower of Hanoi: Number of moves	Tower of Hanoi: Sequence of moves	Chinese rings: Number of moves	Chinese rings: Sequence of moves
1	1	1	1	1
2	$3 = 1 + 2$	1|21	$2 = 2 \times 1$	21
3	$7 = 2 + 5$	12|13121	$5 = 2 \times 2 + 1$	13121
4	$15 = 5 + 10$	12131|2141213121	$10 = 2 \times 5$	2141213121
5	$31 = 10 + 21$		$21 = 2 \times 10 + 1$	
6	$63 = 21 + 42$		$42 = 2 \times 21$	
7	$127 = 42 + 85$		$85 = 2 \times 42 + 1$	
8	$255 = 85 + 170$		$170 = 2 \times 85$	

We see that the moves for a 4-disc Tower of Hanoi game are made up of two parts: the equivalent moves for a 3-ring Meleda game in reverse order, followed by the moves for a 4-ring Meleda. Now, the number of moves for the n-disc Tower of Hanoi is $2^n - 1$ and it is not too hard to guess from the table that the number of moves for the n-ring Meleda is

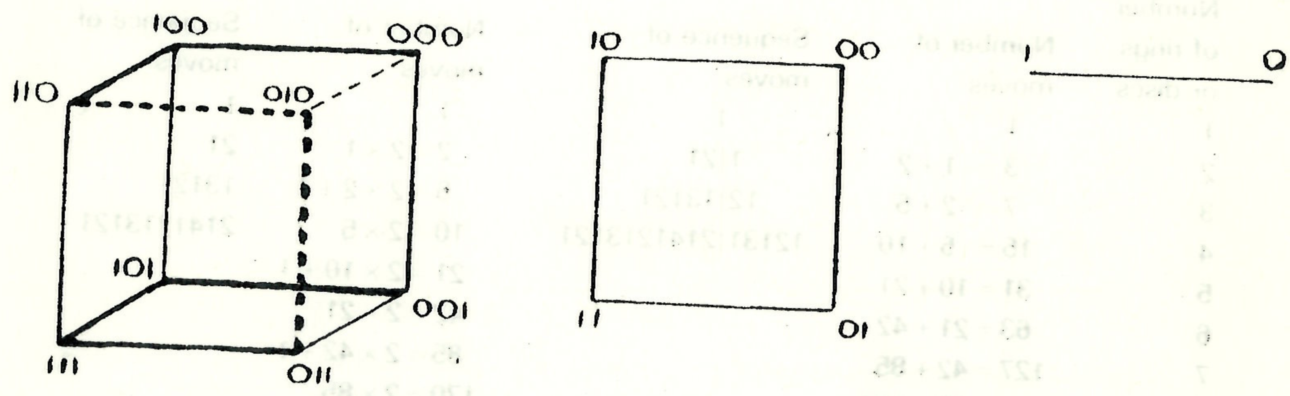
$$\tfrac{1}{3}(2^{n+1} - 2) \text{ for } n \text{ even}, \quad \tfrac{1}{3}(2^{n+1} - 1) \text{ for } n \text{ odd}.$$

(For example, for $n = 8$, $\tfrac{1}{3}(2^{n+1} - 2) = 170$; for $n = 7$, $\tfrac{1}{3}(2^{n+1} - 1) = 85$.)
These results can be proved by induction. Can you do this?

The Gray code

A Gray code is a way of symbolising the counting numbers in a positional notation so that when the numbers are in counting order, any adjacent pair differ in their digits in exactly one place and the absolute difference at that position is 1. (It is really very simple as we will see in a minute.) To appreciate the value of such a system, consider what happens when the odometer of a car reads 9,999 miles. To register the next mile, five wheels must rotate to show 10,000. Because the wheels move slowly, there is little chance of error. But if counting is recorded electronically at enormously high speeds, the likelihood of error increases enormously if two or more digits change simultaneously. The Gray code gives a considerable increase in accuracy in such cases.

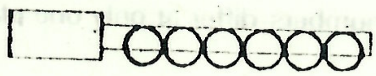
A binary Gray code is based on the digits 0 and 1. If we use only one digit, we can code 2^1 = 2 numbers: 0 → 0 and 1 → 1, where the number on the right is the code for the number on the left. This is represented by the straight line in the illustration below. A Gray code with two digits has 2^2 = 4 numbers: 0 → 00, 1 → 01, 2 → 11, 3 → 10. The corners of a square can be labelled with these numbers, as in the illustration; the labelling is such that the numbers at any pair of adjacent corners differ in only one binary digit. We obtain the code by starting at one corner and visiting all four corners by going around the square. A Gray code with three digits has 2^3 = 8 numbers that can be placed at the corners of a cube as in the illustration. Any path that visits every corner just once generates a Gray code: for example 0 → 000, 1 → 001, 2 → 011, 3 → 010, 4 → 110, 5 → 111, 6 → 101, 7 → 100. As you will no doubt have guessed, binary Gray codes correspond to Hamiltonian circuits on an n-dimensional cube. (See the article on "Binary Magic" cited earlier.)



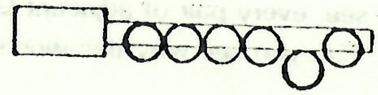
With a large number of digits, there are many possible Gray codes and it becomes important to select one with simple coding and decoding rules. The following rule seems to be the simplest and we shall call it the Gray code from now on. To convert a standard binary number to its Gray code, start with the digit on the right and consider each digit in turn. If the next digit to the left is 0, let the former digit stand. If the next digit to the left is 1, change the former digit. (The digit at the extreme left is assumed to have a 0 on its left, so it remains unchanged.) For example, applying this procedure to the binary number 110111 (= 55) gives the Gray code 101100. To convert back again, consider each digit in turn starting at the right. If the sum of all the digits to the left is even, let the digit stay as it is. If the sum is odd, change the digit. Applying this procedure to 101100 restores the original binary number 110111. The Gray code for the numbers from 0 to 42 is given in the following table. As you can see, every pair of adjacent Gray numbers differ at only one place and, of course, the difference is 1. Can you see why this works?

Number	Binary equivalent	Gray code	Number	Binary equivalent	Gray code
0	0	0	21	10101	11111
1	1	1	22	10110	11101
2	10	11	23	10111	11100
3	11	10	24	11000	10100
4	100	110	25	11001	10101
5	101	111	26	11010	10111
6	110	101	27	11011	10110
7	111	100	28	11100	10010
8	1000	1100	29	11101	10011
9	1001	1101	30	11110	10001
10	1010	1111	31	11111	10000
11	1011	1110	32	100000	110000
12	1100	1010	33	100001	110001
13	1101	1011	34	100010	110011
14	1110	1001	35	100011	110010
15	1111	1000	36	100100	110110
16	10000	11000	37	100101	110111
17	10001	11001	38	100110	110101
18	10010	11011	39	100111	110100
19	10011	11010	40	101000	111100
20	10100	11110	41	101001	111101
			42	101010	111111

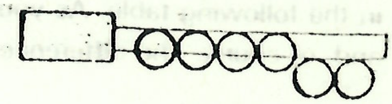
Now let us use the Gray code to solve the Chinese rings puzzle with six rings. Let each ring be represented by a binary digit, 1 if it is on the bar, 0 if it is off. Thus 111111 represents the position with all six rings on the bar. From the table, 111111 is the Gray code for 42, and if we work through the table from 42 back to 0, the successive Gray numbers show which ring is to be removed or put on to solve the puzzle in the minimum number of moves. The first six steps are shown in the illustration below. To solve the puzzle for n rings, we simply convert the Gray number consisting of n ones to a standard binary number to obtain the required number of moves. (In the case of six rings, the Gray number 111111 corresponds to 101010 in standard binary which is 42 in decimal notation.) Can you see why this works? Can you derive the formula for the number of moves given earlier? See if you can use the Gray code to solve the Tower of Hanoi puzzle in the same sort of way.



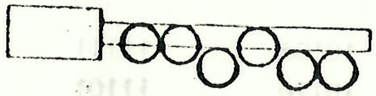
42 = 111111



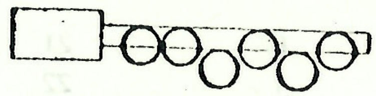
41 = 111101



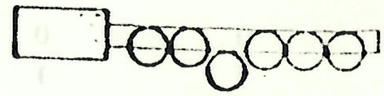
40 = 111100



39 = 110100



38 = 110101



37 = 110111

Further reading

W.W. Rouse Ball and H.S.M. Coxeter, "Mathematical recreations and essays." (University of Toronto Press, 12th edition, 1974.) Pages 316-323.

P. van Delft and J. Botermans, "Creative puzzles of the world." (Cassell, 1978.) Pages 97-102.

Martin Gardner, "Mathematical games." Scientific American, volume 227 (August, 1972), pages 106-109.

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MATHEMATICS (BE IN IT) COMPETITION

Competition Number 5

In higher mathematics and other faraway places, the letters *N, Z, Q, R* and *C* have a peculiar significance. It is usually asserted that

$N = \{0, 1, 2, \ldots\}$ is the natural numbers,

$Z = \{\ldots, -1, 0, 1, 2, \ldots\}$ is the integers,

Q is the rational numbers,

R is the real numbers, and

$C = \{a + b\sqrt{(-1)} : a \text{ and } b \text{ real numbers}\}$ is the complex numbers.

Readers are respectfully requested to give plausible explanations of how these words and symbols really came about. What is natural about the natural numbers? Are there any unnatural numbers? Are they U or non-U? Is *Z* a printer's error for *N*? (Bewahre doch vor Jammerwoch!) Why are the rational numbers called *Q*? Are they sane, intelligent, judicious, or dressed in knickerbockers? Are the real numbers really real, or merely ephemereal?

Prizes of $3, $2 and $1 will be awarded for the best entries. Entries must reach the Editor by 1st January, 1981, and the prize-winners will be announced in Parabola, Volume 17, Number 1. Please make sure that your entry bears your name, year and school.