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THE ART OF COUNTING. I.

B. Musidlak*

Counting is not always as simple as 1, 2, 3, . . . , but, as I hope to show in these articles, it can be a lot more interesting.

Permutations and combinations

The "division principle" is a good starting point. You will all be familiar with the fact that if n distinct objects are to be arranged in a line, this can be done in

$$n! = n(n-1)(n-2)\ldots 2.1$$

different ways. For instance, with three people, any one of them can lead, but once the leader has been chosen, there are two options for the next person and the one bringing up the rear is then determined. Thus, there are six line-ups:

A B C   B A C   C A B
A C B   B C A   C B A

Similarly, in olden times, New South Wales number plates ran from AAA001 to FVZ999, permitting exactly 6.22.26.10.10.10 = 3,432,000 cars to be registered. (Of course, in practice, the number of available combinations was somewhat less than this.) Again, if a teacher distributes r prizes among his n students, the largesse can be assigned in n^r different ways. (What does this become if there is no guarantee that all the prizes will be awarded?)

Things become more interesting when our objects cannot all be distinguished. For instance, if A and A′ are identical twins, the casual observer can detect no difference between

AA′BC and A′ABC,   or   ABA′C and A′BAC,

or any pair of combinations in which A and A′ have been interchanged without disturbing the others. If A, A′, B and C are distinct objects, they can be lined up in 4! = 24 different ways. But if A and A′ are identical twins, we have only 12 = 24/2 line-ups that look different. Again, instead of twins, suppose we have three blue flags, one orange, one white and one green to be placed on six poles to make signals. If the colours were all distinct, there would be 6! signals in which all six flags are used. But if we take one of these configurations and shift the blue flags around, the signal is unaltered. Since there are 3! ways of arranging the three blue flags, the number of different signals (in which the blue flags are indistinguishable) is 6!/3! = 120. How many signals are there if not all the flags need to be used? (This is a little trickier than you might suspect.) In similar vein, if we replaced the white flag by another green one, the number of signals using all six flags would be 120/2 = 60, because interchange of the two green flags would no longer produce a different signal.

* *Bogey Musidlak is a research student in the Department of Pure Mathematics at Macquarie University.*

Binomial coefficients

Usually, in coin-tossing it matters only how many heads and tails appear, the order of appearance being irrelevant. Suppose we want to calculate the number of ways of getting k heads in n tosses of a coin. Equivalently, we can toss n coins once each, so we want the number of arrangements of k heads and n − k tails in a line. We have n! arrangements of the n coins, but shuffling the k heads around in one of these arrangements gives k! identical sequences of heads and tails and, independently, shuffling the n − k tails around gives (n − k)! identical sequences of heads and tails. So the number of distinct ways of obtaining k heads in n tosses is the binomial coefficient

$$C^n_k = n!/k!(n-k)!.$$

In the same way, the number of ways of obtaining a ones, b twos, c threes, d fours, e fives and f sixes in n throws of a die is

$$n!/a!\,b!\,c!\,d!\,e!\,f!\,.$$

(Note that $a+b+c+d+e+f = n$ to make up n throws.)

You should have no difficulty in writing down how many cricket teams (elevens) can be chosen from a squad of fifteen players. How many are there if the captain, vice-captain and wicket-keeper are automatic choices, and only the other eight players have to be selected?

Where else have you run across binomial coefficients? Let us expand

$$(x+y)^n = \underbrace{(x+y)(x+y)\ldots(x+y)}_{n \text{ times}}.$$

We can distribute the brackets one at a time:

$$(x+y)^n = x\underbrace{(x+y)\ldots(x+y)}_{n-1 \text{ times}} + y\underbrace{(x+y)\ldots(x+y)}_{n-1 \text{ times}}$$

$$= x^2\underbrace{(x+y)\ldots(x+y)}_{n-2 \text{ times}} + xy\underbrace{(x+y)\ldots(x+y)}_{n-2 \text{ times}} + yx\underbrace{(x+y)\ldots(x+y)}_{n-2 \text{ times}} + y^2\underbrace{(x+y)\ldots(x+y)}_{n-2 \text{ times}},$$

and so on. We eventually get 2^n terms, each of which looks like $x^{n-r}y^r$ for some r. (For example,

$$(x+y)^3 = x^3 + x^2y + x^2y + xy^2 + x^2y + xy^2 + xy^2 + y^3.)$$

How many times does the term $x^{n-r}y^r$ appear? Now each term is just a sequence of n x's and y's, so we have to count the number of these sequences with exactly r y's and (n − r) x's. This is, fortunately, just the coin-tossing problem again; the answer is C^n_r. So our expansion is

$$(x+y)^n = x^n + C^n_1 x^{n-1}y + C^n_2 x^{n-2}y^2 + \ldots + C^n_r x^{n-r}y^r + \ldots + C^n_{n-1} xy^{n-1} + y^n.$$

(For example, $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.) Note that there is only one way to select a team of n players from a squad of n, so $C^n_n = n!/n!0!$ ought to be 1. For this reason, we make the convention $0! = 1$. The coefficients of x^n and y^n in our binomial expansion can now be written as C^n_0 and C^n_n respectively.

The trinomial expansion of $(x+y+z)^n$ and more general multinomial expansions are just as easy. The terms in the expansion of $(x+y+z)^n$ have the shape $x^a y^b z^c$ with $a+b+c = n$. Such a term can be obtained in $n!/a!b!c!$ different ways, so this is the appropriate trinomial coefficient. No longer will you need to rack your brains over the coefficient of x^{15} in the expansion of $(3+2x^2+3x^5)^{11}$! First, observe that there are two ways to obtain a term in x^{15} from $3^a (2x^2)^b (3x^5)^c$, namely $3^5 (2x^2)^5 (3x^5)^1$ and $3^8 (2x^2)^0 (3x^5)^3$; the multinomial coefficients are $11!/5!5!1! = 2772$ and $11!/8!0!3! = 165$ respectively. So x^{15} has the coefficient $2772.3^5.2^5.3 + 165.3^8.2^0.3^3 = 93894471$. With a minute amount more work, you will be able to find the coefficient of x^{17} or x^{20}. Can you?

The binomial expansion gives some useful identities if we substitute special values for x and y. For instance,

$$2^n = (1+1)^n = C_0^n + C_1^n + C_2^n + \ldots + C_{n-1}^n + C_n^n$$

and

$$0 = (1-1)^n = C_0^n - C_1^n + C_2^n - \ldots + (-1)^n C_n^n$$

Problems

1. In how many ways can you give change of 20 cents? That is, in how many ways can you pay 20 cents using 1, 2, 5, 10 and 20 cent coins?

2. In how many ways can you put the necessary stamps in one row on an ordinary letter sent inside Australia using 2, 5, 10, 15 and 20 cent stamps? The postage at the time of writing is 22 cents.

3. Suppose you have a set of eight weights of 1, 1, 2, 5, 10, 10, 20 and 50 grams respectively. In how many ways can 78 grams be composed of these weights?

4. In a certain house of assembly, there are 21 seats and 3 parties. In how many different ways can the seats be distributed among the parties so that no party attains a majority against a coalition of the other two parties.

5. Show that $C_r^n = C_{r-1}^{n-1} + C_r^{n-1}$. (This shows that the binomial coefficients can be calculated using the Pascal triangle.)

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DON'T DO THAT AGAIN

Here is a sum from the complete lazy man's guide to arithmetic. (Don't try it in an exam!)

$$\frac{1\not6\not6\not6\not6\not6\not6\not6\not6\not6\not6}{\not6\not6\not6\not6\not6\not6\not6\not6\not6\not64} = \frac{1}{4}$$

It works no matter how many 6's you use. Can you find another horrible example?