MUCH ADO ABOUT NOTHING

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The harmonic series,

$$S = 1 + 2^{-1} + 3^{-1} + 4^{-1} + \ldots$$

is divergent, that is if we add enough terms together, we can produce a partial sum which is as large as we like. (Can you prove this? You will find three different arguments in the solution to problem 434 later in this issue.) One of the points dramatically illustrated by the articles on "Bricks that almost topple over" (Parabola, Volume 15, Number 2) is that the series diverges rather slowly. For example, it takes a little over 1.5×10^{43} terms to produce a partial sum exceeding 100. If we estimate, rather optimistically that a computer takes one microsecond to add each term, and imagine that our computer has been running since the beginning of the universe about 10^{10} years ago, then the computer will only have reached a partial sum of about 40 as you read these lines.

Suppose we now thin out the harmonic series by dropping all the reciprocals of integers whose decimal representations contain one or more zeros. That is, we consider the series

$$T = 1 + 2^{-1} + 3^{-1} + \ldots + 9^{-1} + 11^{-1} + \ldots + 19^{-1} + 21^{-1} + \ldots.$$

Surprisingly, the series now converges. To remove some of the surprise, we can observe that most integers with a large number of digits have zeros in their decimal expansions. In fact, there are $9 \times 10^{n-1}$ integers with n digits, but only 9^n without zeros, so the proportion of n-digit numbers without zeros is $(9/10)^{n-1}$ which tends to 0 as n tends to infinity. To show that T converges, we group the terms by taking first the reciprocals of the 1-digit integers, then the reciprocals of the 2-digit integers without zeros, and so on. The reciprocals of the 1-digit integers are all less than or equal to 1 and there are 9 of them, so they contribute at most 9 to T; the reciprocals of the 2-digit integers without zeros are all less than 1/10 and there are 9^2 of them, so they contribute at most $9^2/10$ to T. Continuing in this way, we see that

$$T < 9 + 9^2/10 + 9^3/10^2 + \ldots = 90.$$

(The series here is a geometric series with common ratio 9/10.)

What more can we say about this mysterious sum T? Suppose we naively try to compute the sum by adding up the successive terms. Let T(k) be the partial sum obtained by adding the reciprocals of all the integers of at most k digits with no zeros. The first few values of T(k) are shown in the table below. It is clear from the fourth column of the table that, even after the first half million terms, we are nowhere near the limit of the series. In fact, the remainder after all the

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reciprocals of the integers of at most k digits have been dealt with consists of 9^{k+1} reciprocals of (k + 1)-digit numbers each exceeding 10^{-k-1}, then 9^{k+2} reciprocals of (k + 2)-digit numbers each exceeding 10^{-k-2}, and so on. Thus the remainder after we reach the partial sum T(k) is

$$T - T(k) > (9/10)^{k+1} + (9/10)^{k+2} + \ldots = 10(9/10)^{k+1}.$$

k	Partial sums T(k)	Number of terms in T(k)	Contribution of k-digit integers U(k) = T(k) − T(k − 1)	Ratio U(k)/U(k − 1)
1	2.82897	9	2.82897	—
2	4.89448	90	2.06551	0.73
3	6.71873	819	1.82425	0.88
4	8.35751	7380	1.63878	0.898
5	9.83213	66429	1.47462	0.8998
6	11.1593	597870	1.32717	0.90000

The number of terms in the partial sum T(k) is

$$9 + 9^2 + \ldots + 9^k > 9^k.$$

For example, if we add all the reciprocals of integers with at most 13 digits without zeros, the remainder exceeds $10(9/10)^{14} > 2$ and we have added more than 9^{13} terms. Allowing an optimistic microsecond per term, we would need more than a month of computing time even to get this rather woeful result.

We cannot calculate T by adding terms until the remainders are small. What we have to do instead is to calculate the remainders with sufficient accuracy to enable us to extrapolate the partial sums in the table above. To do this, let U(k) be the sum of the reciprocals of the k-digit numbers without zeros. Our table suggests that U(k)/U(k − 1) tends rapidly to 9/10 as k tends to infinity. Assuming this for the moment, we can calculate T as follows. First, we make the approximations

$$U(k+1) = (9/10)U(k),$$

$$U(k+2) = (9/10)U(k+1) = (9/10)^2U(k),$$

and so on. Then we take

$$\begin{aligned} T &= T(k-1) + (T(k) - T(k-1)) + (T(k+1) - T(k)) + \ldots \\ &= T(k-1) + U(k) + U(k+1) + U(k+2) + \ldots \\ &= T(k-1) + U(k)\{1 + 9/10 + (9/10)^2 + \ldots\} \\ &= T(k-1) + 10\,U(k). \end{aligned}$$

In particular, if we take k = 6 and use the values of T(5) and U(6) from the table above, we get T = 23.10343.

What do you think of this answer? You might care to check the figures given in the table. How far can these computations be taken before round-off error in the machine becomes significant? Do your calculations support the contention that U(k)/U(k − 1) approaches 9/10? Use the formula T = T(k − 1) + 10 U(k) to estimate T for some other values of k. How do your answers compare with the one given above?

Can you prove that U(k)/U(k − 1) tends to 9/10 as k tends to infinity? Actually you will need to do a little more in order to see how close we have got to the value of T. Try to estimate how quickly U(k)/U(k − 1) approaches 9/10. Now we should be getting somewhere. If we can estimate the differences

$$U(k+1) - (9/10)\,U(k), \quad U(k+2) - (9/10)^2\,U(k), \ldots,$$

then we will be able to estimate the error T − T(k − 1) − 10 U(k) in the value of T. Can you do this? To what accuracy is it possible to calculate T by this method?

The harmonic series may converge when its digits are thinned out in other ways. You may like to try your hand at the following problems.

(i) Estimate the sum of the reciprocals of all the integers whose decimal representations contain only odd digits.

(ii) Estimate the sum of the reciprocals of all the integers whose decimal digits are all different.

(iii) Estimate the sum of the reciprocals of all the integers whose decimal digits, reading from left to right, form a non-increasing sequence, for example 875532220.

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A TRYING PROBLEM

Here is a problem to help while away those idle moments on a Saturday afternoon before the big game starts. Neanderthal Fred is about to convert a try made in the corner, that is, he wants to kick for goal from some point P on the touch line as shown in the diagram. A reasonable first guess is that Fred should choose the point P so that the angle θ subtended by the goal at P is as large as possible. Can you calculate the distance x so that the angle θ is maximised. How does your answer compare with the practice in real-life football? What other factors do you think should be taken into account in choosing the best position for the point P?