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Problem 80.3. We shall consider some problems involving the roots α, β and γ, say, of the cubic equation

$$x^3 + qx + r = 0. \tag{1}$$

A recent 4 unit paper (1978) asked for the cubic equation with the roots α^2, β^2 and γ^2, where α, β and γ are the given roots of (1). We can solve this problem as follows. Let the required cubic be

$$(x - \alpha^2)(x - \beta^2)(x - \gamma^2) \equiv x^3 + ax^2 + bx + c.$$

By comparing the coefficients of like powers of x on the two sides of this equation, we obtain expressions for the coefficients a, b and c in terms of α, β and γ, namely

$$a = -(\alpha^2 + \beta^2 + \gamma^2), \; b = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2, \; c = -\alpha^2\beta^2\gamma^2.$$

Similarly, from (1), $(x - \alpha)(x - \beta)(x - \gamma) \equiv x^3 + qx + r$, so

$$\alpha + \beta + \gamma = 0, \; \alpha\beta + \beta\gamma + \gamma\alpha = q, \; \alpha\beta\gamma = -r. \tag{2}$$

Now, after a little bit of algebra,

$$a = -(\alpha^2 + \beta^2 + \gamma^2) = -(\alpha + \beta + \gamma)^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 2q,$$

$$b = \alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) = q^2$$

and

$$c = -\alpha^2\beta^2\gamma^2 = -r^2.$$

So the required equation is

$$x^3 + 2qx^2 + q^2x - r^2 = 0. \tag{3}$$

This routine method will generally solve this type of problem, but there are some tricks available to expedite the calculations. For example, if we set $y = x^2$ in (1), we have $(y + q)x + r = 0$, that is

$$x = -r/(y + q).$$

Now, if we substitute in (1) and multiply through by $(y + q)^3/r$, we get the cubic equation

$$(y + q)^3 - q(y + q)^2 - r^2 = 0. \tag{4}$$

Since the roots of (1) are $x = \alpha, \beta$ and γ, the roots of this new equation are $y = \alpha^2, \beta^2$ and γ^2. Of course, (3) and (4) are identical equations as you can check by removing the brackets.

These problems allow plenty of scope for fast algebraic foot-work. We give three more examples. Suppose, again, that α, β and γ are the roots of (1). We will find the cubic equations whose roots are

(i) α^3, β^3 and γ^3;

(ii) $\alpha^3 + \beta^3$, $\beta^3 + \gamma^3$ and $\gamma^3 + \alpha^3$; and

(iii) α^5, β^5 and γ^5.

Here are some possible methods for obtaining the answers.

(i) Since α, β and γ are the roots of (1), we have

$$\alpha^3 = -q\alpha - r, \ \beta^3 = -q\beta - r, \ \gamma^3 = -q\gamma - r,$$

and, from (2), we find

$$\alpha^3 + \beta^3 + \gamma^3 = -q(\alpha+\beta+\gamma) - 3r = -3r,$$

$$\begin{aligned}\alpha^3\beta^3 + \beta^3\gamma^3 + \gamma^3\alpha^3 &= (q\alpha+r)(q\beta+r) + (q\beta+r)(q\gamma+r) + (q\gamma+r)(q\alpha+r)\\ &= q^2(\alpha\beta+\beta\gamma+\gamma\alpha) + 2qr(\alpha+\beta+\gamma) + 3r^2\\ &= q^3 + 3r^2,\end{aligned}$$

and

$$\alpha^3\beta^3\gamma^3 = -r^3,$$

so the required equation is

$$(x^3 + 3rx^2 + (q^3+3r^2)x + r^3 = 0. \quad (5)$$

Alternatively, we can set $z = x^3$, so that $z = -qx - r$. If we substitute $x = -(z+r)/q$ in (1) and multiply through by $-q^3$, we get the equation

$$(z+r)^3 + q^3(z+r) - q^3 r = 0, \quad (6)$$

whose roots are $z = \alpha^3$, β^3 and γ^3. This is only a lightly disguised version of (5).

(ii) From the working in (i),

$$\alpha^3 + \beta^3 = -3r - \gamma^3, \beta^3 + \gamma^3 = -3r - \alpha^3, \gamma^3 + \alpha^3 = -3r - \beta^3,$$

so we want the equation whose roots are $-3r-\alpha^3$, $-3r-\beta^3$ and $-3r-\gamma^3$.

Put $y = -3r - z$, so that $z = -3r - y$, and substitute in (6):

$$(2r+y)^3 + q^3(2r+y) + q^3 r = 0,$$

that is

$$y^3 + 6ry^2 + (q^3+12r^2)y + (3q^3r+8r^3) = 0.$$

Since the roots of (6) are $z = \alpha^3$, β^3 and γ^3, this last equation has the roots $y = -3r-\alpha^3$, $-3r-\beta^3$ and $-3r-\gamma^3$, as required. Alternatively, we can do this without using (i) by substituting $x = (2r+y)/q$ in (1). Can you see why?

(iii) First observe that

$$\alpha^5 = \alpha^2(-q\alpha - r) = -q\alpha^3 - r\alpha^2 = qr + q^2\alpha - r\alpha^2.$$

Using this, and similar expressions for β^5 and γ^5, we calculate

$$\alpha^5 + \beta^5 + \gamma^5 = 3qr + q^2(\alpha+\beta+\gamma) - r(\alpha^2+\beta^2+\gamma^2) = 3qr + 2qr = 5qr,$$

$$\alpha^5\beta^5 + \beta^5\gamma^5 + \gamma^5\alpha^5 = 3q^2r^2 + 2q^3r(\alpha+\beta+\gamma) - 2qr^2(\alpha^2+\beta^2+\gamma^2) + q^4(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$- q^2r(\alpha\beta^2+\beta\gamma^2+\gamma\alpha^2+\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha) + r^2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)$$

$$= 3q^2r^2 + 4q^2r^2 + q^5 - 3q^2r^2 + q^2r^2 = q^5 + 5q^2r^2,$$

and
$$\alpha^5\beta^5\gamma^5 = -r^5,$$

so the required equation is $x^3 - 5qrx^2 + (q^5 + 5q^2r^2)x + r^5 = 0$. (The calculation of $\alpha^5\beta^5 + \beta^5\gamma^5 + \gamma^5\alpha^5$ is a little messy, but you should not find it too hard to fill in the missing steps.)

Now try your hand at the following problems. (The solutions will be given in the next issue.)

(i) Let α, β and γ be the roots of $x^3 + qx + r = 0$. Find the cubic equations whose roots are

(a) $\alpha^2 + \beta^2$, $\beta^2 + \gamma^2$ and $\gamma^2 + \alpha^2$; and

(b) $1 + 1/\alpha$, $1 + 1/\beta$ and $1 + 1/\gamma$.

(ii) Solve completely the simultaneous equations

$$a + b + c = 0$$
$$a^3 + b^3 + c^3 = 3$$
$$a^5 + b^5 + c^5 = -10\,.$$

Problem 80.1 (continued). Recall that the problem asked for all real numbers x satisfying

$$|x-1| < |x+1|. \qquad (7)$$

The earlier discussion in Parabola, Volume 16, Number 1, solved this problem algebraically: by squaring both sides of (7) and simplifying, we find that (7) is equivalent to $x > 0$. Now, here are two geometrical solutions.

First, observe that $|x-1|$ is just the distance between x and 1 along the real line and $|x+1|$ is the distance between x and -1. So x satisfies (7) if and only if x is closer to 1 than it is to -1. The midpoint of the interval from -1 to 1, namely 0, is equidistant from -1 and 1, so again we see that (7) is equivalent to $x > 0$.

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