LETTERS TO THE EDITOR

A BIRTHDAY SURPRISE

Dear Sir,

Do you think that there would be two people in your class whose birthdays fall on the same day and month? Would you be prepared to bet on it? Let us consider the chances of having n people whose birthdays fall on n different days. Now the years are not of equal length and the birth rates are not quite constant throughout the year. However, to a first approximation, we can take a random selection of people as equivalent to a random selection of birthdays and consider the year as consisting of 365 days. Then there are 365^n equally likely ways of assigning the birthdays of n people. If these birthdays are to be distinct, we have 365 choices for the first one, $365-1$ choices for the second, $365-2$ choices for the third, and so on. So the probability that all n birthdays are different is

$$p(n) = 365 \,.\, 364 \,.\, 363 \,.\, \ldots \,(365-n+1)/(365)^n.$$

The probability that two of the n people have a common birthday is $1-p(n)$. For $n = 30$, this works out to be 0.71, which makes it an almost certain bet. You might even be able to find someone to take you up on it.

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Editor's comments.

Paul's birthday odds are astounding at first sight. You might like to try out the experiment in your class, to see how normal you are. The formula for p(n) looks a bit forbidding, but it is easy to get a good approximation when n is fairly small. We can write

$$p(n) = (1-1/365)(1-2/365)(1-3/365)\ \ldots\ (1-(n-1)/365)$$

If we multiply this out and neglect all the cross products, we get the rough approximation

$$p(n) \approx 1 - (1+2+\ldots+(n-1))/365 = 1 - n(n-1)/730.$$

For $n = 10$, this gives the value 0.877, while the correct value of p(10) is 0.883. Is this approximation any good when $n = 30$? For larger n, we get a better approximation by taking logarithms and using the approximation $\log(1-x) \approx -0.4343x$. (Our logarithms are to the base 10.) This gives

$$\begin{aligned}\log p(n) &= \log(1-1/365) + \log(1-2/365) + \ldots + \log(1-(n-1)/365)\\ &\approx -0.4343\,(1+2+3+\ldots+(n-1))/365\\ &\approx -0.000595\, n(n-1).\end{aligned}$$

For $n = 30$, this gives the value 0.304, while the correct value of $p(30)$ is 0.294.

How big does n have to be to ensure that $p(n) < \frac{1}{2}$? In a group of this size, the probability that at least two people have a common birthday exceeds $\frac{1}{2}$.

Here is another problem. A lift starts with 7 passengers and stops at 10 floors. What is the probability that no two passengers leave at the same floor? You will find more surprises of this sort in our series of articles on the art of counting which begins in this issue.

MEET PI AGAIN

Dear Sir,

Richard Wilson and Professor Prokhovnik (Parabola, Volume 15, Number 2, and Volume 14, Number 3) have shown how π can be obtained as the limit of various trigonometric expressions. It is amazing how often π turns up, even in limits of quite elementary functions. For example, consider the integral

$$\int_0^1 \sqrt{(1-x^2)}\, dx = \tfrac{1}{4}\pi,$$

which gives the area of the part of the unit circle in the first quadrant. Put $f(x) = \sqrt{(1-x^2)}$. If we approximate the integral by Simpson's rule, dividing the interval $[0,1]$ into $2n$ subintervals each of width $h = 1/2n$, we see that

$$(h/3)\ \{f(0) + 4f(h) + 2f(2h) + 4f(3h) + \ldots + 4f(1-3h) + 2f(1-2h) + 4f(1-h) + f(1)\}$$

tends to $\frac{1}{4}\pi$ as $n \to \infty$. This formula gives the following approximations to π:

n	approximation to π
1	2.98
10	3.136
100	3.1414

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Editor's comments.

Kieran's letter reminded me of some brilliant analysis by John Wallis, an English mathematician who lived from 1616 to 1703. His work had a considerable influence on Newton and on the discovery of the calculus, but today he is chiefly remembered for "Wallis's product". Wallis considered the integral

$$I(p,k) = \int_0^1 (1-x^{1/p})^k\, dx,$$

which gives the area inside the part of the "ellipse" $x^{1/p} + y^{1/k} = 1$ in the first quadrant. It is not too hard to compute the value of $I(p,k)$ when k is a small non-negative integer: for example,

$$I(p,0) = \int_0^1 1.dx = 1,$$

$$I(p,1) = \int_0^1 (1 - x^{1/p})\,dx = 1/(p+1),$$

$$I(p,2) = \int_0^1 (1 - x^{1/p})^2\,dx = 2/(p+1)(p+2).$$

A bit more work serves to confirm the guess that if k is a non-negative integer, then

$$I(p,k) = 1.\,2.\,3.\,\ldots\,.k/(p+1)(p+2)\ldots(p+k). \qquad (1)$$

It seems that Wallis obtained this formula by extrapolation from Table 1 which shows the values of $1/I(p,k)$ for various p and k. In fact, Table 1 is a version of Pascal's triangle: any number in the table is the sum of the two numbers next to it, one above and the other to the left. Note that

p \ k	0	1	2	3	4	5	. . .	10
0	1	1	1	1	1	1		1
1	1	2	3	4	5	6		11
2	1	3	6	10	15	21		66
3	1	4	10	20	35	56		286
4	1	5	15	35	70	126		1001
⋮								
10	1	11	66	286	1001	3003		184756

Table 1

$$I(p,k) = (k/(p+k))I(p,k-1). \qquad (2)$$

(If you know the formula for integration by parts, you should be able to use it to obtain (2) and so verify (1).) Now Wallis has the bold idea of interpolating the entries in Table 1 to obtain the values of $I(p,k)$ when k is a multiple of ½; Table 2 shows the values of $1/I(p,k)$ which he obtained. Let us

p \ k	0	1/2	1	3/2	2	5/2	3	7/2
0	1	1	1	1	1	1	1	1
1/2	1	□	3/2	4□/3	15/8	8□/5	35/16	64□/35
1	1	3/2	2	5/2	3	7/2	4	9/2
3/2	1	4□/3	5/2	8□/3	35/8	64□/15	105/16	125□/21
2	1	15/8	3	35/8	6	63/8	10	99/8
5/2	1	8□/5	7/2	64□/15	63/8	128□/15	231/16	512□/35
3	1	35/16	4	105/16	10	231/16	20	429/16

Table 2 (□ = $4/\pi$)

examine the row with $p = \frac{1}{2}$. We already know that $I(\frac{1}{2},0) = 1$ and, from (2), we obtain successively

$$I(\tfrac{1}{2},1) = (2/3)I(\tfrac{1}{2},0) = 2/3, \quad I(\tfrac{1}{2},2) = (4/5)I(\tfrac{1}{2},1) = 8/15,$$

and, in general,

$$I(\tfrac{1}{2},k) = \frac{2k}{2k+1} I(\tfrac{1}{2},k-1) = \frac{2k(2k-2)\ldots 6.4.2}{(2k+1)(2k-1)\ldots 7.5.3} \quad \text{for } k = 0,1,2,\ldots$$

Now, how do we get the values of $I(1/2,1/2)$, $I(1/2,3/2)$, ... that go in between $I(\frac{1}{2},0)$, $I(\frac{1}{2},1)$, $I(\frac{1}{2},2)$, ...? If we can find one of them, we can use the recurrence relation (2) to get the others one by one. It just so happens that $I(\frac{1}{2},\frac{1}{2})$ is the area inside the part of the circle $x^2 + y^2 = 1$ in the first quadrant, that is $I(\frac{1}{2},\frac{1}{2}) = \frac{1}{4}\pi$. So we have

$$I(1/2,3/2) = (3/4)\, I(1/2,1/2) = 3\pi/16, \quad I(1/2,5/2) = (5/6)\, I(1/2,3/2) = 15\pi/96.$$

and, in general,

$$I(\tfrac{1}{2},k+\tfrac{1}{2}) = \frac{2k+1}{2k+2} I(\tfrac{1}{2},k-\tfrac{1}{2}) = \frac{(2k+1)(2k-1)\ldots 5.3}{(2k+2)\,2k\ldots 6.4} \cdot \frac{\pi}{4} \quad \text{for } k = 0,1,2,\ldots.$$

We can fill in the rest of Table 2 by making use of the symmetry. Note that $I(p,k)$ is the area inside the curve $x^{1/p} + y^{1/k} = 1$ in the first quadrant. If we interchange p and k, we merely reflect this curve in the line $y = x$ which does not change the area. (See Figure 1.) Thus $I(k,p) = I(p,k)$. In particular, we now know all the values $I(p,\frac{1}{2}) = I(\frac{1}{2},p)$, so the recurrence relation (2) gives us

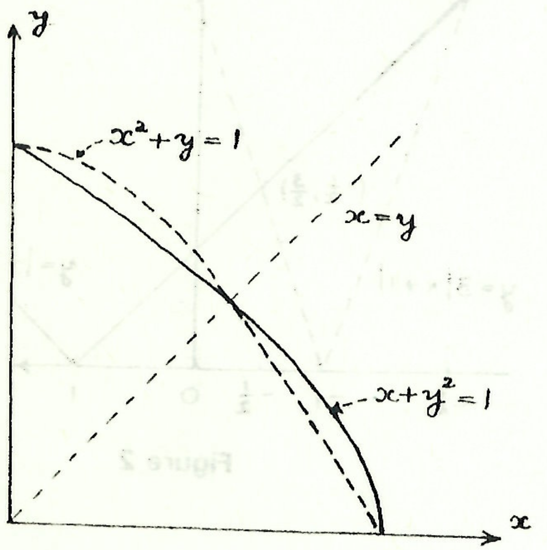

Figure 1

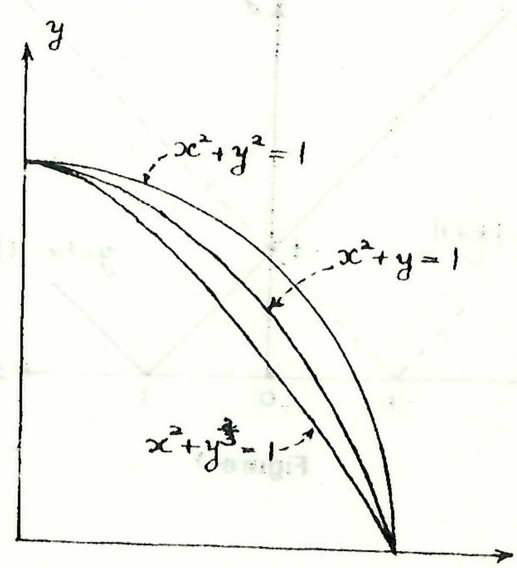

Figure 2

$I(p,3/2)$, $I(p,5/2)$, ... in a line stretching out to th'crack of doom, although for convenience, we have stopped at $I(p,7/2)$. Let us now compare the areas in Figure 2. If we fix p and let k get larger, the area $I(p,k)$ gets smaller. In particular,

$$I(\tfrac{1}{2},k) < I(\tfrac{1}{2},k-\tfrac{1}{2}) < I(\tfrac{1}{2},k-1)$$

for any positive integer k. If we use the values of these integrals obtained above and do a little rearranging, we get the inequalities

$$\left[\frac{4.6.8.\ldots(2k-2)2k}{3.5.7.\ldots(2k-3)(2k-1)}\right]^2 \frac{2}{2k+1} < \frac{\pi}{4} < \left[\frac{4.6.8.\ldots(2k-2)2k}{3.5.7.\ldots(2k-3)(2k-1)}\right]^2 \frac{2}{2k}$$

which can be used to estimate π as closely as we like. We can even let k tend to infinity to get Wallis's product

$$\tfrac{1}{4}\pi = \lim_{k\to\infty} \frac{2.4.4.6.6.\ldots(2k-2)2k.2k}{3.3.5.5.7.\ldots(2k-1)(2k-1)(2k+1)}.$$

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H.S.C. Corner (continued from page 25)

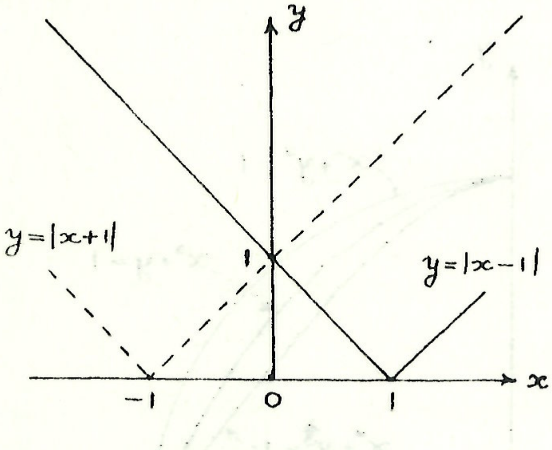

Figure 1

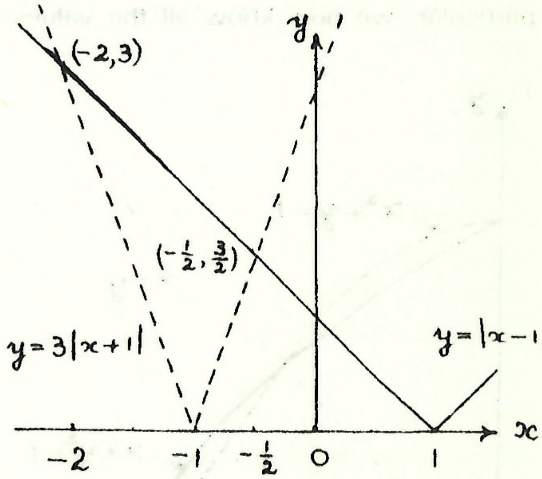

Figure 2

Another method uses the graphs of $|x-1|$ and $|x+1|$. These are drawn in Figure 1; clearly, (7) holds if and only if x is positive. To convince you how good this idea is, let us consider the inequality

$$|x-1| < 3\,|x+1|. \qquad (8)$$

From Figure 2, (8) holds if $x < -2$ or $x > -\tfrac{1}{2}$. The intersections of the two graphs are obtained by solving $-(x-1) = 3(x+1)$ and $-(x-1) = -3(x+1)$ and this is the only work required.