1980 SCHOOL MATHEMATICS COMPETITION

Junior Division

1. Algernon announced that on his birthday this year his age would be equal to the sum of the digits of the year in which he was born. When was he born?

Bertie blurted out that on *his* birthday in 1985 his age would be equal to the sum of the digits of the year in which *he* was born. Prove that this is impossible.

Find all the years in the twentieth century which are like 1985 in this respect.

Solution. Algernon's maximum possible age is 27, the sum of the digits of 1899, so he must have been born in the twentieth century. Suppose that his year of birth is $1900 + 10x + y$, where x and y are integers from the set $\{0,1,2,\ldots,9\}$. Algernon's assertion amounts to

$$1980 - 1900 - 10x - y = 1 + 9 + x + y, \quad \text{that is } 11x + 2y = 70.$$

The only admissible solution with x and y in the set $\{0,1,2,\ldots,9\}$ is $x = 6$, $y = 2$. So Algernon was born in 1962.

In the same way, Bertie's broadside becomes

$$1985 - 1900 - 10x - y = 1 + 9 + x + y, \quad \text{that is } 11x + 2y = 75,$$

but this equation has no solutions with x and y in the set $\{0,1,2,\ldots,9\}$.

Now suppose the year is $1900 + N$ and Clarence claims that his age is equal to the sum of the digits in the year, say $1900 + 10x + y$, in which he was born. Then x and y satisfy

$$11x + 2y = N - 10. \qquad (1)$$

Consider the numbers $N-10, N-12, N-14, \ldots, N-30$ which are the values of $N - 10 - 2y$ as y runs through $0,1,2,\ldots,10$. We have here 11 numbers and no two of them give the same remainder on division by 11; indeed if 11 divides $(N-2y) - (N-2y') = 2(y'-y)$, then 11 divides $y'-y$ and so $y' = y$ since y and y' both come from the set $\{0,1,2,\ldots,10\}$. It follows that exactly one of these numbers is divisible by 11. Consequently, the equation (1) has exactly one solution in integers x and y with $0 \leqslant y \leqslant 10$. Now Clarence's claim is possible if the equation (1) has a solution in integers x and y with $0 \leqslant x, y \leqslant 9$. Clarence's claim is impossible if the solution of (1) that we have found requires $y = 10$, that is, whenever $N - 10 - 2.10 = N - 30$ is a multiple of 11. This occurs for

$$N = 8, 19, 30, 41, 52, 63, 74, 85 \text{ and } 96. \qquad (2)$$

Clarence's claim is also impossible if our solution of (1) requires $x < 0$; this occurs for

$$N = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 15, 17 \text{ and } 19. \qquad (2')$$

However, this is not quite the end of the matter because Clarence could have been born in the nineteenth century, say in $1800 + 10x + y$. In this case we have to solve the equation

$$1900 + N - 1800 - 10x - y = 1 + 8 + x + y, \text{ that is } 11x + 2y = N + 91,$$

in integers x and y with $0 \leqslant x, y \leqslant 9$. As before, this equation has exactly one solution with $0 \leqslant y \leqslant 10$. This solution is inadmissible if either $N + 71$ is divisible by 11, that is

$$N = 6, 17, 28, 39, 50, 61, 72, 83, 94, \qquad (3)$$

because then $y = 10$, or if

N is even and greater than 26, or odd and greater than 15, (3')

because then $x \geqslant 10$. Finally we see that Clarence's claim is impossible if N is in one of the lists (2), (2') and also in one of the lists (3), (3'). The corresponding years $1900 + N$ are

1906, 1917, 1919, 1930, 1941, 1952, 1963, 1974, 1985 and 1996. ☆

2. A four digit number and its square end in the same four digits. Find the number.

Solution. We have to find a four-digit number N such that $N^2 - N = N(N-1)$ is divisible by $10^4 = 2^4 5^4$. In order to achieve this, one of N and $N-1$ must be divisible by 2^4 and the other by 5^4; the latter is an odd multiple of 5^4 and less than 10000, so it must be one of the numbers

625, 1875, 3125, 4375, 5625, 6875, 8125, 9375.

We examine the numbers which are 1 more or 1 less than the numbers in this list to see whether any is a multiple of 16, and we find that 16 divides 624 and 9376. These correspond to $N = 625$ and $N = 9376$. We discard the first of these since it is not a proper four-digit number. This leaves $N = 9376$ as the only solution. ☆

3. For any two numbers x and y, the symbol $x \triangle y$ denotes another number determined by x and y. The operation $\triangle$ has the following two properties:

(i) $0 \triangle x = x$ for all x, and

(ii) $(x \triangle y) \triangle z = z \triangle (xy) + x \triangle z + y \triangle z - 2z$ for all x,y,z.

Evaluate $8 \triangle 9$.

Solution. First,

$$\begin{aligned} y\triangle z &= (0\triangle y)\triangle z, \quad \text{by (i)},\\ &= z\triangle 0 + 0\triangle z + y\triangle z - 2z, \quad \text{by (ii)},\\ &= z\triangle 0 + z + y\triangle z - 2z, \quad \text{by (i)}\\ &= y\triangle z + z\triangle 0 - z, \end{aligned}$$

so (iii) $z \triangle 0 = z$ for all z.

Now $$x\triangle y = (x\triangle y)\triangle 0, \quad \text{by (iii)},$$
$$= 0\triangle(xy) + x\triangle 0 + y\triangle 0 - 0, \quad \text{by (ii)},$$
$$= xy + x + y, \quad \text{by (i) and (iii)}.$$

This shows that the only meaning for $x\triangle y$ is $x\triangle y = xy + x + y$, but we still have to check that (i) and (ii) are consistent by showing that they hold when $x\triangle y = xy + x + y$. This is easy:

$$0\triangle x = 0.x + 0 + x = x \text{ for all } x, \quad \text{and}$$
$$(x\triangle y)\triangle z = (xy + x + y)z + (xy + x + y) + z$$
$$= (zxy + z + xy) + (xz + x + z) + (yz + y + z) - 2z$$
$$= z\triangle(xy) + x\triangle z + y\triangle z - 2z \quad \text{for all } x,y,z.$$

Hence $x\triangle y = xy + x + y$ and, in particular, $8\triangle 9 = 89$. ☆

4. A certain eight-digit number is divisible by 101. Prove that if the last digit is moved to the front, the new number is again divisible by 101.

Solution. Let N be the given number and a be its last digit.
The new number is M (say) $= 10^7a + (N-a)/10$, so

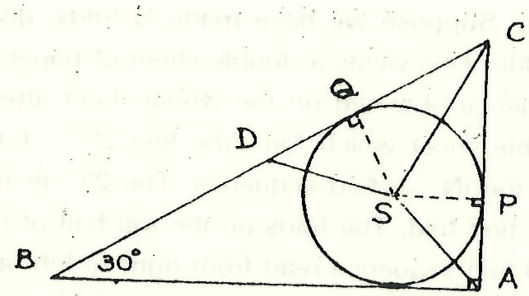
$$10M = N + (10^8 - 1)a. \tag{4}$$

Now,

$$10^8 - 1 = (10^4 - 1)(10^4 + 1) = (10^2 + 1)(10^2 - 1)(10^4 + 1),$$

so $10^8 - 1$ is a multiple of 101. Also, 10M is divisible by 101 if and only if M is. Consequently, (4) shows that if N is divisible by 101, then so is M. ☆

5. Let ABC be a right-angled triangle with hypotenuse BC and with angle ABC equal to 30°. Let S be the centre of the inscribed circle, touching the three sides of the triangle as shown, and let D be the mid-point of BC. Show that AS = DS.



Solution. First, note that AC = ½BC = DC. To see this, reflect the triangle ABC in the side AB; this gives an equilateral triangle and AC is a half of one of its sides. Alternatively, in the right-angled triangle ABC, AC = BC sin 30° = ½BC.

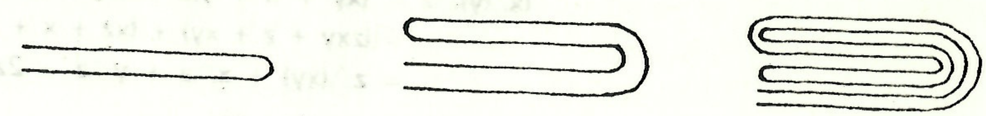
Next, note tht CS bisects the angle ACB. To see this, consider the triangles CPS and CQS. The angles CPS and CQS are right-angles, the side CS is common, and the sides PS and QS are equal because they are both radii of the inscribed circle which touches AC and BC at P and Q respectively. Thus the triangles CPS and CQS are congruent and angle PCS equals angle QCS.

Now consider the triangles ACS and DCS. From what we have already said, the sides AC and DC are equal, the angles ACS and DCS are equal, and the side CS is common. So the triangles ACS and DCS are congruent and, in particular, AS = DS. ☆

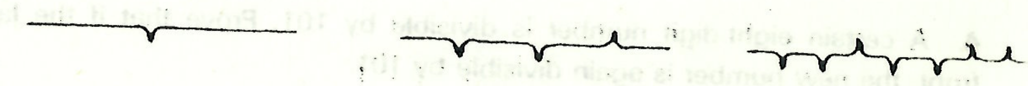
6. A piece of paper is folded repeatedly, right half over left. When the paper is unfolded, a sequence of valleys and ridges is obtained. Label valleys as 1 and ridges as 0. For example, the patterns produced by the first three folds are as follows:

1 fold — 2 folds — 3 folds

paper folded



paper unfolded



	1 fold	2 folds	3 folds
sequence	1	110	1101100

Show that each additional fold gives a longer sequence whose first "half" is just the sequence formed at the preceding step.

Find the 150th, 151st and 152nd terms of the sequence formed when the folding is continued indefinitely.

Does this sequence ever have four consecutive 0's?

Solution. Suppose we have made N folds, giving a sequence with $2^N - 1$ terms. Unfold the last $N - 1$ folds. This yields a double sheet of paper on which the sequence of folds is just the same as the sequence obtained on the whole sheet after $N - 1$ folds. Thus the folds on the bottom half of the double sheet which form the first $2^{N-1} - 1$ terms of the N-fold sequence, are the same as the folds in the $(N - 1)$-fold sequence. The 2^{N-1}-th term of the N-fold sequence is the valley formed at the very first fold. The folds on the top half of the double sheet, which form the last $2^{N-1} - 1$ terms of the N-fold sequence read from right to left, are the same as the folds on the bottom half. When the initial fold is unfolded, the folds on the top sheet are turned upside-down, so the n-th fold of the N-fold sequence is the opposite of the $(2^N - n)$-th fold for $1 \leqslant n \leqslant 2^{N-1} - 1$. This gives the following simple rule for generating the sequence: to get the N-fold sequence, take the $(N - 1)$-fold sequence, adjoin a 1, and then write down the $(N - 1)$-fold sequence in reverse order and with its 0's replaced by 1's and its 1's replaced by 0's. For example

4 folds: 1101100 – 1 – 1100100

5 folds: 110110011100100 – 1 – 110110001100100

The 8-fold sequence has 255 terms and the 150th, 151st and 152nd are just the opposites of the 106th, 105th and 104th respectively, since these groups are symmetrically placed about the central term. In the 7-fold sequence, the 106th, 105th and 104th terms are the opposites of the 22nd, 23rd and 24th respectively and from the 5-fold sequence above, these are all 0's. So the 150th, 151st and 152nd terms of the sequence are all 0's.

The N-fold sequence, formed according to the rule given above, has the following shape

$$\underbrace{1\ 1\ 0\ \ldots\ 1\ 0\ 0}_{(N-1)\text{-fold sequence}} - 1 - \underbrace{1\ 1\ 0\ \ldots\ 1\ 0\ 0}_{(N-1)\text{-fold sequence reversed}}$$

The quadruples in the N-fold sequence arise from the following sources:

(i) the quadruples in the (N – 1)-fold sequence;

(ii) the reversals of the quadruples in the (N – 1)-fold sequence with 0's and 1's interchanged, and

(iii) the quadruples 1001, 0011, 0111 and 1110 coming from the "middle" of the N-fold sequence.

We can therefore get all the quadruples in the sequence by taking N = 4 and listing all the quadruples occurring in (i), (ii) and (iii), together with their reversals:

1101	1011	0110	1100	0011	0111	1110
0100	0010	1001	(1100)	(0011)	0001	1000

(The quadruples in the second row are the reversals of those in the first row with 0's and 1's interchanged.) In particular, the sequence never has four consecutive 0's. ☆

Senior Division

1. A four-digit number and its square end in the same four digits. Find the number.

Solution. See Junior Division, question 2. ☆

2. A set of n integers is chosen at random from the set {1,2,3,...,49,50} of the first 50 positive integers. What is the minimum value of n which ensures that the chosen set always contains a pair of integers, one of which is three times the other?

Solution. Partition the integers 1,2,3,...,50 into groups with common ratio 3 as follows:

(i) {1,3,9,27}, {2,6,18}, {4,12,36}, {5,15,45};

(ii) {7,21}, {8,24}, {10,30}, {11,33}, {13,39}, {14,42}, {16,48}; and

(iii) {17}, {19}, {20}, {22}, {23}, {25}, {26}, {28}, {29}, {31}, {32}, {34}, {35}, {37}, {38}, {40}, {41}, {43}, {44}, {46}, {47}, {49}, {50}.

In any selection from these integers in which no integer selected is three times another, we can have at most two elements from each of the sets in the list (i), and at most one from each of the sets in the list (ii), and there are no restrictions on the choice of elements from the sets in the list (iii). In this way, we can choose 2.4 + 7 + 23 = 38 integers so that none is three times any other. Moreover, in any choice of 39 integers from among 1,2,3,...,50, we must have a pair of integers, one of which is three times the other. ☆

3. Let n be a non-zero integer. Show that n can be written as the difference of two integer squares in only a finite number of ways.

Let a, b, c and d be integers. Show that the equation $x^2 + ax + b = y^2 + cy + d$ has infinitely many solutions in integers x and y if and only if $a^2 - 4b = c^2 - 4d$.

Solution. Suppose $n = x^2 - y^2$ for some integers x and y. Then $n = (x+y)(x-y)$, so $x+y$ and $x-y$ are divisors of n. Consider one such factorisation, say $x+y = r$, $x-y = s$ with $rs = n$. Now $x = \frac{1}{2}(r+s)$, $y = \frac{1}{2}(r-s)$ are uniquely determined by r and s. Since n has only a finite number of integer divisors, there are only finitely many possible choices for x and y.

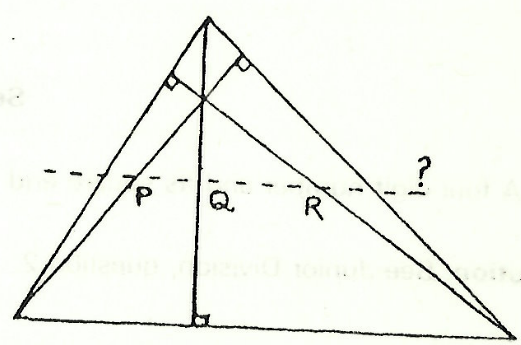
Complete the square on both sides of the given equation:

$$(x + \tfrac{1}{2}a)^2 + b - \tfrac{1}{4}a^2 = (y + \tfrac{1}{2}c)^2 + d - \tfrac{1}{4}c^2.$$

After a little rearrangement, this becomes

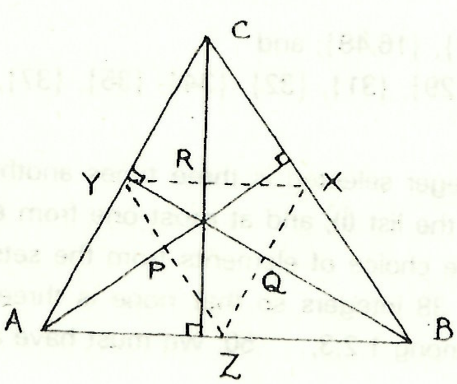
$$(2x+a)^2 - (2y+c)^2 = (a^2-4b) - (c^2-4d). \qquad (5)$$

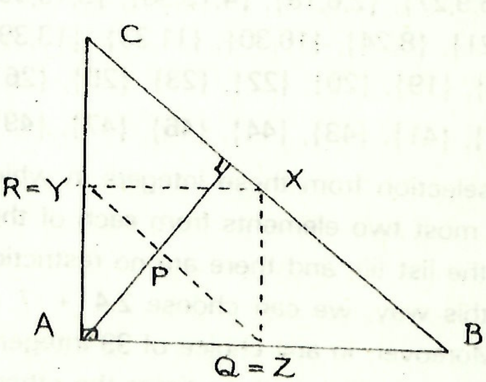
Put $n = (a^2-4b) - (c^2-4d)$. If $n = 0$, then a and c have the same parity and so we can get infinitely many integer solutions of (5) by taking $2(x-y) = c-a$ with $y = 0, \pm 1, \pm 2, \ldots$. If $n \neq 0$, then (5) has only finitely many integer solutions by the first part of the question. ☆



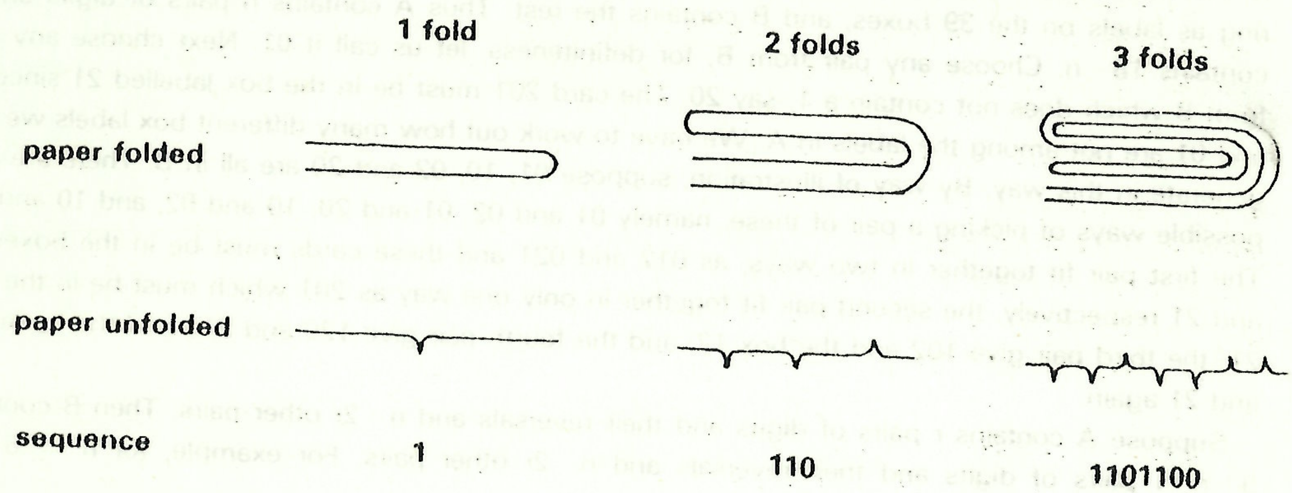
4. Suppose that the midpoints P, Q and R, of the three altitudes of a triangle fall on a line. Show that one of the angles of the triangle must be a right-angle.

Solution. In the figure, X, Y and Z are the midpoints of the three sides of the triangle. By several applications of the midpoint theorem that the line joining the midpoint of two sides of a triangle is parallel to the third side, we see that P lies on YZ, Q lies on ZX and R lies on XY. Thus, P, Q and R lie on different sides of the triangle XYZ and so cannot be collinear, unless two of the points P, Q and R coincide with vertices of the triangle XYZ. This only happens when one of the angles of the triangle ABC is a right-angle. ☆





5. A piece of paper is folded repeatedly, right half over left. When the paper is unfolded, a sequence of valleys and ridges is obtained. Label valleys as 1 and ridges as 0. For example, the patterns produced by the first three folds are as follows:



Show that each additional fold gives a longer sequence whose first "half" is just the sequence formed at the preceding step.

Find the 150th, 151st and 152nd terms of the sequence formed when the folding is continued indefinitely.

Does this sequence ever have four consecutive 0's?

Which of the 16 possible quadruples, 0000, 0001, 0010, 0011, . . ., 1111, can occur in the sequence?

Solution. See Junior Division, question 6. ☆

6. You are given 1000 cards, numbered from 000 to 999, and 100 boxes, numbered 00 to 99, and instructed to place each card in a box whose number can be obtained from the number of the card by deleting one of its digits. For example, the card 123 must be placed in one of the three boxes 12, 13, or 23.

Show that all the cards can be distributed, according to this rule, using only 50 of the boxes.

Prove that any distribution which satisfies the rule uses at least 50 boxes.

Solution. To accommodate the cards 000, 111, 222, . . ., 999, we need the ten boxes 00, 11, 22, . . ., 99. These will also accommodate any card with a repeated digit. Consequently, we need to show that the cards labelled with distinct digits can be placed in just 40 boxes and that at least 40 boxes are required no matter how the distribution is done.

Now any card labelled with three distinct digits has either two odd digits or two even digits, so we can accommodate all these cards in the boxes ab where $a \neq b$ and $a \equiv b \pmod 2$. There are exactly 40 such boxes since there are ten possible choices for a and each of these can be paired with four possible values of b. Thus 40 boxes suffice for the cards with distinct digits.

Suppose that we have somehow packed the cards with distinct digits into 39 boxes. The labels on these boxes contain 78 digits in all, so at least one of the digits 0, 1, 2, ..., 9 occurs fewer than eight times on these labels. For definiteness, suppose the digit 0 occurs only n times, with $n \leq 7$. We divide the pairs 0a, a0 with a = 1, 2, ..., 9 into two sets: A contains the pairs occurring as labels on the 39 boxes, and B contains the rest. Thus A contains n pairs of digits and B contains $18 - n$. Choose any pair from B; for definiteness, let us call it 01. Next choose any pair from B which does not contain a 1, say 20. The card 201 must be in the box labelled 21 since 20 and 01 are not among the labels in A. We have to work out how many different box labels we can generate in this way. By way of illustration, suppose 01, 10, 02 and 20 are all in B. There are four possible ways of picking a pair of these, namely 01 and 02, 01 and 20, 10 and 02, and 10 and 20. The first pair fit together in two ways, as 012 and 021 and these cards must be in the boxes 12 and 21 respectively, the second pair fit together in only one way as 201 which must be in the box 21; the third pair give 102 and the box 12; and the fourth pair give 120 and 210 and the boxes 12 and 21 again.

Suppose A contains r pairs of digits and their reversals and $n - 2r$ other pairs. Then B contains $9 - n + r$ pairs of digits and their reversals and $n - 2r$ other pairs. For example, for $n = 6$ and $r = 2$, we might have

$$A = \{06,70\} \cup \{08,80,09,90\}, \quad B = \{01,10,02,20,03,30,04,40,05,50\} \cup \{60,07\}.$$

Consider the different ways of choosing a pair of pairs from B, as described above. There are $\frac{1}{2}(9-n+r)(8-n+r)$ choices like 01,02 from the first part of B and each gives two boxes, that is $(9-n+r)(8-n+r)$ boxes altogether. There are $(9-n+r)(n-2r)$ choices like 10,60 with the first pair from the first part of B and the second pair from the second part of B and both of the shape a0 or both of the shape 0a, and each gives two boxes, that is $2(9-n+r)(n-2r)$ boxes altogether. There are $\frac{1}{2}(n-2r)(n-2r-1)$ choices like 60,07 from the second part of B and each gives at least one box, so at least $\frac{1}{2}(n-2r)(n-2r-1)$ boxes altogether. Finally, there are $\frac{1}{2}r(r-1)$ choices like 08,09 from the second part of A and each of these gives two boxes. For example, if we choose 08 and 09, and any digit, say 1, not occurring in A, then the cards 189, 198, 891 and 981 require two boxes not previously considered. There are always enough digits not occurring in A to avoid duplications, so this gives $r(r-1)$ boxes altogether. If we include the n boxes in A that we first thought of, we now have a grand total of at least

$$n + (9-n+r)(8-n+r) + 2(9-n+r)(n-2r) + \tfrac{1}{2}(n-2r)(n-2r-1) + r(r-1)$$
$$= 72 + (3/2)n - \tfrac{1}{2}n^2 + 2nr - 19r$$

boxes. For fixed $n \leq 8$, this is a decreasing function of r over the range $0 \leq r \leq \frac{1}{2}n$. So we can take $r = \frac{1}{2}n$ and find that the number of boxes is at least

$$72 + (3/2)n - \tfrac{1}{2}n^2 + n^2 - 19(\tfrac{1}{2}n) = 72 - 8n + \tfrac{1}{2}n^2.$$

This is a decreasing function of n in the interval $0 \leq n \leq 8$ and its value at $n = 8$ is 40. Bravo! Thus at least 40 boxes are required for the cards with distinct digits. In fact, the scheme we gave at the beginning is essentially the only way to do the distribution in 40 boxes.

Your editor admits that this is not a particularly artistic solution and that the magic 40 should pop out with more élan, but he has not been able to make it do so. Perhaps you can help. ☆