# PROBLEM SECTION

***You are warmly invited to submit solutions to one or more of the problems that follow. Please begin each problem on a new page and make sure that each answer bears your name, year and school. Solutions together with the names of successful solvers will be published in the issue after next.***

***To spur on your efforts, remember that the best and most consistent solvers this year will receive book prizes. If some of the problems seem a little obscure at first reading, may we offer this advice: "Don't panic!" Happy puzzling.***

**455.** The rule for leap years runs as follows: A year which is divisible by 4 is a leap year except that years which are divisible by 100 are not leap years unless they are divisible by 400. Now, January the first, 1980, was a Tuesday. Prove that January the first in a new century (that is, 1900, 2000, 2100, . . .) is never a Tuesday. What days of the week can it be?

**456.** Find the last two digits of $9^{10}$ in decimal notation. Find the last three digits of $9^{9^{9}}$.

**457.** Find all pairs of positive numbers x,y such that $(x^2+4y^2)/xy = 5$ and $x^y = 4$.

**458.** A recent archeological find contains a partly obliterated description of a rather astonishing ancient Persian relay race between two teams of two members each in which a handicap was arrived at by the following process. Each of the four runners took a handful of wheat. After pairing off into teams, each team counted its combined wheat grains into x cups (the number x is no longer decipherable), each cup containing the same number of grains, until there were too few grains remaining to again place one in each cup. If a team was left with r grains, it did not start running until the count of r after the starting signal. In one such event, the four athletes Artaxerxes, Belshazzar, Cyrus and Darius (henceforth referred to by their initials) took handfuls of wheat containing 8965, 8434, 7672 and 8512 grains respectively. It is reported that C said to A: "I suggest we pair together as a team, since then, you see, there would be no grains of wheat left over. We would start right on the signal." To this, A replied: "Unfortunately, a minute ago, B addressed me in identical terms and I accepted his offer." In the ensuing race, in spite of being considerably handicapped by having to start on the count of y (also obliterated), C and D emerged victorious. Can you decide what were the obliterated numbers x and y?

**459.** For an integer n written in decimal notation, we define the number S(n) to be the sum of the units, hundreds, ten thousands, . . . digits of n, plus ten times the sum of the tens, thousands, hundred thousands, . . . digits. For example, $S(123456789) = (9+7+5+3+1) + 10(8+6+4+2) = 225$. Show that $n - S(n)$ is always a multiple of 99. Find $S(S(S(1980^{1980})))$.

460. At a party, there are 2n people, of whom each one is acquainted with at least n others. Prove that it is possible to seat them at a circular table so that each is seated between two acquaintances.

461. Let p(1), p(2), . . ., p(n) be the first n primes. Partition these primes into two sets A and B, so that A consists of some of the primes and B consists of all the others. Let a be the product of the primes in A and let b be the product of the primes in B. It is claimed that $m = |a-b|$ is always either 1 or a prime. Is this true? If it is true, prove it. If not, find the smallest n for which the claim is false and the smallest composite number m obtainable in this way.

462. You are given five weights of identical appearance weighing respectively 1, 2, 3, 4, and 5 grams. Show how, with an accurate beam balance but no other standard weights, it is possible to sort out which is which in only five weighings.

463. In the preceding question, show that you cannot guarantee to identify the weights in only four weighings.

464. A number of circles are drawn in the plane. There are exactly 12 points in the plane which lie on more than one circle. What is the smallest possible number of regions into which the plane could be subdivided by these circles?

465. Let AB be a line segment whose midpoint M is marked, and let P be a point in the plane not on AB. Show how, using no drawing instrument except a straight-edge (and a pencil), you can construct a line through P parallel to AB.

466. Given a point P and three distances x, y and z, show how to construct an equilateral triangle ABC, using only straight-edge and compass, so that P is inside the triangle and PA = x, PB = y and PC = z.

## SOLUTIONS TO PROBLEMS FROM VOLUME 15, NUMBER 3

429. Let a be a positive integer. Prove that the fraction $(a^3+2a)/(a^4+3a^2+1)$ is in its lowest terms.

**Solution:**

Apply Euclid's algorithm to the numbers $a^4 + 3a^2 + 1$ and $a^3 + 2a$:

$$a^4 + 3a^2 + 1 = a(a^3+2a) + (a^2+1)$$

$$a^3 + 2a = a(a^2+1) + a$$

$$a^2 + 1 = a.a + 1.$$