460. At a party, there are $2n$ people, of whom each one is acquainted with at least n others. Prove that it is possible to seat them at a circular table so that each is seated between two acquaintances.

461. Let $p(1), p(2), \ldots, p(n)$ be the first n primes. Partition these primes into two sets A and B, so that A consists of some of the primes and B consists of all the others. Let a be the product of the primes in A and let b be the product of the primes in B. It is claimed that $m = |a-b|$ is always either 1 or a prime. Is this true? If it is true, prove it. If not, find the smallest n for which the claim is false and the smallest composite number m obtainable in this way.

462. You are given five weights of identical appearance weighing respectively 1, 2, 3, 4, and 5 grams. Show how, with an accurate beam balance but no other standard weights, it is possible to sort out which is which in only five weighings.

463. In the preceding question, show that you cannot guarantee to identify the weights in only four weighings.

464. A number of circles are drawn in the plane. There are exactly 12 points in the plane which lie on more than one circle. What is the smallest possible number of regions into which the plane could be subdivided by these circles?

465. Let AB be a line segment whose midpoint M is marked, and let P be a point in the plane not on AB. Show how, using no drawing instrument except a straight-edge (and a pencil), you can construct a line through P parallel to AB.

466. Given a point P and three distances x, y and z, show how to construct an equilateral triangle ABC, using only straight-edge and compass, so that P is inside the triangle and $PA = x$, $PB = y$ and $PC = z$.

SOLUTIONS TO PROBLEMS FROM VOLUME 15, NUMBER 3

429. Let a be a positive integer. Prove that the fraction $(a^3+2a)/(a^4+3a^2+1)$ is in its lowest terms.

Solution:

Apply Euclid's algorithm to the numbers $a^4 + 3a^2 + 1$ and $a^3 + 2a$:

$$a^4 + 3a^2 + 1 = a(a^3+2a) + (a^2+1)$$
$$a^3 + 2a = a(a^2+1) + a$$
$$a^2 + 1 = a.a + 1.$$

Using the above equations in turn, we see that the common divisors of $a^4 + 3a^2 + 1$ and $a^3 + 2a$ are also divisors of $a^2 + 1$, then of a, and finally of 1. Hence $a^4 + 3a^2 + 1$ and $a^3 + 2a$ are relatively prime and the fraction $(a^3+2a)/(a^4+3a^2+1)$ is in its lowest terms.

Solutions along these lines were received from D. Everett (Kotara High School), K. Lim (St. Ignatius' College), P. Rider (St. Leo's College), S. Tolhurst (Springwood High School) and O. Wright (Davidson High School).

J. Tually (Sydney Grammar) observed that a^4+3a^2+1 differs from a multiple of a by 1 and also differs from $(a^2+2)(a^2+1)$ by 1, whence it must be relatively prime to both a and a^2+2, and so to their product a^3+2a.

Incidentally, note that the polynomials $x^3 + 2x = x(x^2+2)$ and $x^4 + 3x^2 + 1$ are relatively prime, but this does not automatically imply that their values at $x = a$ are relatively prime for every a. For example, the polynomials $x+2$ and x^2+2 are relatively prime but their values at $x = 2$, namely 4 and 6, are not.

430. Let p be a prime greater than 3. Show that p^2 is one more than a multiple of 12.

Solution from S. Tolhurst (Springwood High School) and R. Wilson (The King's School).

If p is a prime greater than 3, then it must have one of the forms $12n+1$, $12n+5$, $12n+7$, or $12n+11$, where n is a positive integer. Squaring each of these expressions gives

$$(12n+1)^2 = 144n^2 + 24n + 1 \equiv 1 \pmod{12},$$

$$(12n+5)^2 = 144n^2 + 120n + 25 \equiv 1 \pmod{12},$$

$$(12n+7)^2 = 144n^2 + 168n + 49 \equiv 1 \pmod{12},$$

$$(12n+11)^2 = 144n^2 + 264n + 121 \equiv 1 \pmod{12}.$$

Thus, when p is a prime greater than 3, we have $p^2 \equiv 1 \pmod{12}$.

Correct solutions were also received from A. Buchanan (Sydney Grammar), D. Everett (Kotara High School), K. Lim (St. Ignatius' College), J. Tually (Sydney Grammar), O. Wright (Davidson High School) and R. Youhana (North Sydney Boys' High School).

431. Adjoin to the digits 632. . . three more digits so that the resulting six digit number is divisible by each of 7, 8 and 9.

Solution:

Since 7, 8 and 9 are relatively prime, a number that is divisible by 7, 8 and 9 is also divisible by $7 \times 8 \times 9 = 504$. Now $504 \times 1254 = 632016$ and $504 \times 1255 = 632520$ are the only multiples of 504 between 632000 and 633000, so the only sets of 3 digits that can be adjoined to 632. . . to form a multiple of 7, 8 and 9 are . . .016 and . . .520.

Solutions along these lines were received from T. Abberton (St. Paul's College, Bellambi), D. Everett (Kotara High School), A. Jenkins (North Sydney Boys' High School), K. Lim (St. Ignatius' College), J. Tually (Sydney Grammar), S.S. Wadhwa (Ashfield Boys' High School), R. Wilson (The King's School), O. Wright (Davidson High School) and R. Youhana (North Sydney Boys' High School).

432. Find all three digit numbers which are equal to the sum of the factorials of their digits.

Solution:

The 3-digit number cannot contain a digit equal to 7, 8 or 9 because 7!, 8! and 9! have more than 3 digits. Neither can 6 be included because 6! = 720 and our number cannot contain a digit greater than or equal to 7. Thus only the digits 0 to 5 inclusive can be used. The greatest possible "factorial sum" for the 3-digit number is therefore 5! + 5! + 5! = 360, so our number must be less than this. If the number exceeds 168, it must contain two 5's, so the only possible candidates greater than 168 are 255 and 355, neither of which has the required property. On the other hand, our 3-digit number is at least 100, so its "factorial sum" must be at least 100 and it must contain at least one 5. The possible candidates remaining are

$$125, 135, 145, 150, 151, 152, 153 \text{ and } 154.$$

By trial, the only number with the required property is 145.

Solutions along these lines were received from D. Everett (Kotara High School), P. Rider (St. Leo's College), S. Tolhurst (Springwood High School), J. Tually (Sydney Grammar), S.S. Wadhwa (Ashfield Boys' High School), R. Wilson (The King's School) and O. Wright (Davidson High School).

433. Which is larger, 100^{300} or 300!?

Solution I from D. Everett (Kotara High School) and K. Lim (St. Ignatius' College):

In the diagram,

$$\log(300!) = \log 2 + \log 3 + \log 4 + \ldots + \log 300$$

is given by the sum of the areas of the rectangles. The sum is approximately equal to but slightly larger than the area under the curve $y = \log x$ between $x = 1$ and $x = 300$, that is

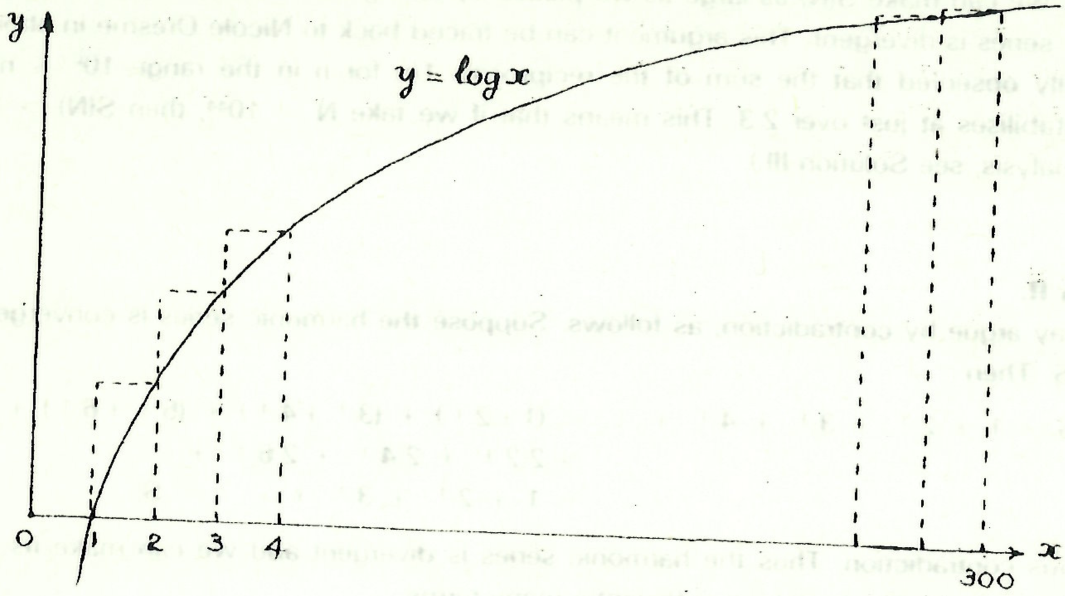
$$\log(300!) > \int_1^{300} \log x \, dx = [x \log x - x]_1^{300} = 1412.1\ldots$$

(These logarithms are all natural logarithms to the base e.)

On the other hand,

$$\log(100^{300}) = 300 \log 100 = 1381.5\ldots$$

Consequently, $\log(300!) > \log(100^{300})$ and $300! > 100^{300}$.



Solution II:

300! =

3060575122164406360353704612972686293885888041735769994167767412594765
3317671686746551527142247757334993914788870172636886426390775900315422
6842927906974559841225476930271954604008012215776252176854255965356903
5067887252643218962642993652045764488303889097539434896254360532259807
7652127082243763944912012867867536830571229368194364995646049816645022
7716500185176546469340112226034729724066333258583506870150169794168850
3537521375549102891264071571548302822849379526365801452352331569364822
3343679925459409527682060806223281238738388081704960000000000000000000
0000000000000000000000000000000000000000000000000000000

J. Tually (Sydney Grammar) found 3 efficient ways of using his calculator to settle this problem.

434. Show that if N is taken sufficiently large, the sum $1+2^{-1}+3^{-1}+4^{-1}+\ldots+N^{-1}$ is larger than 100. (See the articles on "Bricks that almost topple" in Parabola, Volume 14, Number 3 and Volume 15, Number 2.)

Solution I:

Let $S(N) = 1 + 2^{-1} + 3^{-1} + 4^{-1} + \ldots + N^{-1}$. The reciprocals of the 1-digit numbers (in base 10) all exceed 1/10 and there are 9 of them, so they contribute more than 9/10 to S(N). The reciprocals of the 2-digit numbers all exceed 1/100 and there are 90 of them, so they also contribute more than 9/10. And so on. Let us take $N = 10^{112}$. Then we can apply our argument 112 times and it follows that $S(10^{112}) > 112 \times 9/10 > 100$.

In fact, we can make S(N) as large as we please by taking N to be sufficiently large, that is, the harmonic series is divergent. This argument can be traced back to Nicole Oresme in about 1350.

J. Tually observed that the sum of the reciprocals $1/n$ for n in the range $10^k \leqslant n < 10^{k+1}$ quickly stabilises at just over 2.3. This means that if we take $N = 10^{44}$, then $S(N) > 100$. (For a further analysis, see Solution III.)

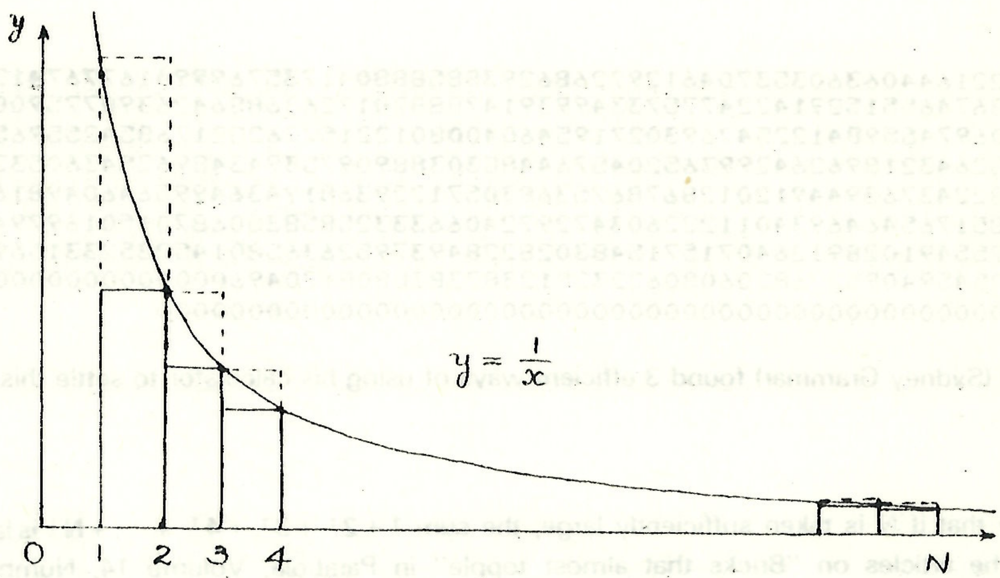
Solution II:

We may argue by contradiction, as follows. Suppose the harmonic series is convergent and call its sum S. Then

$$\begin{aligned} S = 1 + 2^{-1} + 3^{-1} + 4^{-1} + \ldots &= (1 + 2^{-1}) + (3^{-1} + 4^{-1}) + (5^{-1} + 6^{-1}) + \ldots \\ &> 2.2^{-1} + 2.4^{-1} + 2.6^{-1} + \ldots \\ &= 1 + 2^{-1} + 3^{-1} + \ldots = S, \end{aligned}$$

an obvious contradiction. Thus the harmonic series is divergent and we can make its partial sum S(N) as large as we like by taking sufficiently many terms.

Solution III:



From the diagram, the area under the curve $y = 1/x$ from $x = 1$ to $x = N$ is greater than $2^{-1} + 3^{-1} + \ldots + N^{-1}$, which is the sum of the areas of the rectangles drawn in solid lines, and the difference between these two areas increases with N. Thus

$$\gamma(N) = \log N - S(N) + 1 = \int_1^N x^{-1}\, dx - (2^{-1} + 3^{-1} + \ldots + N^{-1})$$

increases with N. (All our logarithms are natural logarithms to the base e.) Moreover, the area under the curve $y = 1/x$ from $x = 1$ to $x = N$ is less than $1 + 2^{-1} + \ldots + (N-1)^{-1}$, which is the sum of the areas of the rectangles drawn in dashed lines, so $\gamma(N) < 1$. It follows from all this that $\gamma(N)$ approaches a limit as $N \to \infty$, and so too does $S(N) - \log N$. The last limit is usually denoted by γ and is called Euler's constant; the first few decimal digits of γ are 0.5772156649. . . . If we take N sufficiently large, then S(N) will be approximately $\log N + \gamma$ and $\log N$ will be large, so we can make S(N) as large as we please. To see how large we need to take N in order to make S(N) larger than 100, we need to know how quickly $S(N) - \log N$ approaches its limit γ. From the discussion on bricks that almost topple in Parabola, Volume 15, Number 2,

$$S(N) = \log N + \gamma + 1/2N - \epsilon(N)$$

where $0 < \epsilon(N) < 1/12N^2$, so we have

$$S(N) > \log N + \gamma \geq 100$$

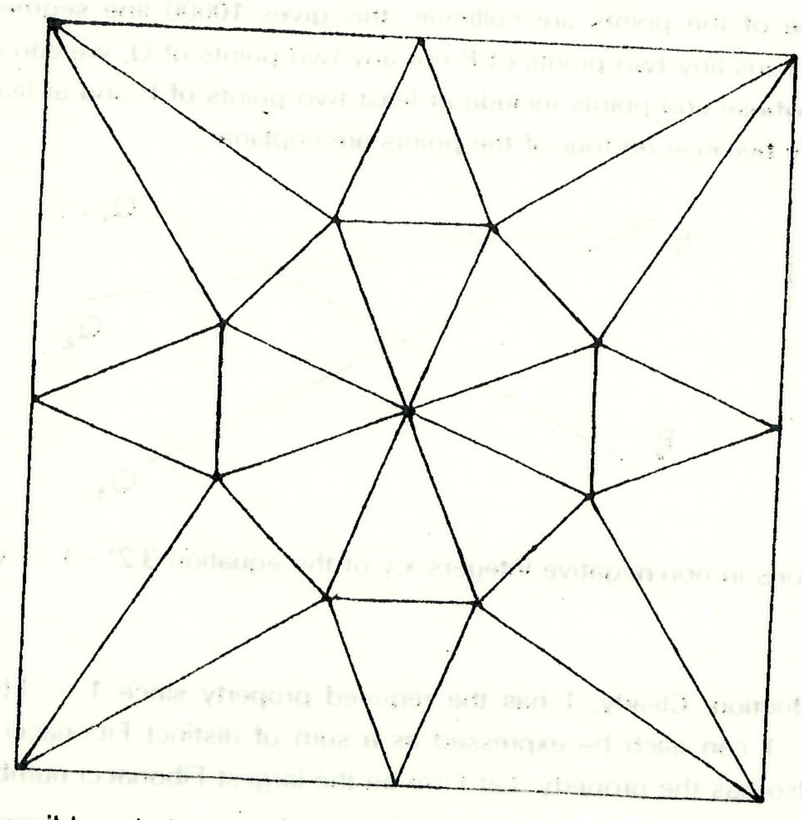
if $\log N > 100 - \gamma$, that is $N > e^{100-\gamma} \approx 1.5 \times 10^{43}$. As you can see, this argument actually gives the exact number of terms required to make S(N) exceed 100.

Correct solutions were received from D. Everett (Kotara High School), K. Lim (St. Ignatius' College), R. Wilson (The King's School) and R. Youhana (North Sydney Boys' High School).

435. Show how to cut a square piece of paper into acute-angled triangles.

Solution from K. Svendsen (Busby High School):

Consider the following diagram:



There are many possible solutions to this problem.

438. Prove that every positive integer can be written as the sum of distinct Fibonacci numbers. (The Fibonacci numbers F_n are defined by $F_1 = 1$, $F_2 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$).

Solution:

Since $y^2 - 1 = (y+1)(y-1) = 3.2^x$, we can write

$$y \pm 1 = 3.2^a, \quad y \mp 1 = 2^b,$$

where $a + b = x$. Now the difference between the two factors is 2, so $\pm 2 = 2^a(3 - 2^{b-a})$, whence $2^a = 1$ or 2.

First case. Suppose $2^a = 1$. Then $a = 0$, $3 - 2^{b-a} = 2$ and consequently $b = 0$. Thus $x = 0 + 0$ and $y^2 = 4$.

Second case. Suppose $2^a = 2$. Then $a = 1$, $3 - 2^{b-a} = \pm 1$ and consequently $b = 2$ or 3. Thus $x = 3$, $y^2 = 25$, or $x = 4$, $y^2 = 49$.

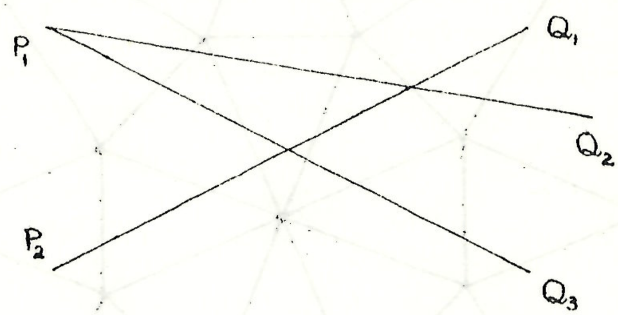
The solutions are therefore $(x,y) = (0, \pm 2)$, $(3, \pm 5)$ and $(4, \pm 7)$.

A correct solution was received from J. Tually (Sydney Grammar).

437. Two hundred points are distributed in space so that no three are collinear and no four are coplanar. Prove that it is possible to draw 10,000 line segments joining them without completing a single triangle.

Solution from K. Svendsen (Busby High School):

Select any 100 points and call this collection P. Let Q be the collection consisting of the remaining 100 points. Now join each of the 100 points of P by a line segment to each of the 100 points of Q. Since no three of the points are collinear, this gives 10000 line segments. Since none of these line segments joins any two points of P nor any two points of Q, we can only form a triangle by using segments whose end points include at least two points of P and at least two points of Q. But this is impossible because no four of the points are coplanar.



436. Find all solutions in non-negative integers x,y of the equation $3.2^x + 1 = y^2$.

Solution:

We shall use induction. Clearly, 1 has the required property since $1 = F(1)$. Assume that the numbers $1, 2, \ldots, k-1$ can each be expressed as a sum of distinct Fibonacci numbers. We must now show that k also has the property. Let $F(m)$ be the largest Fibonacci number not larger than k, that is

$$F(m) \leqslant k < F(m+1) = F(m) + F(m-1).$$

Then $k = F(m) + \ell$, where $\ell < F(m-1)$. By the induction hypothesis, ℓ can be written as a sum of distinct Fibonacci numbers, obviously all less than $F(m-1)$. Hence k has the required property, and the proof is complete.

Correct solutions were received from A. Buchanan (Sydney Grammar), D. Everett (Kotara High School), K. Lim (St. Ignatius' College), K. Svendsen (Busby High School) and R. Wilson (The King's School).

439. Given any two people, we may classify them as friends, enemies, or strangers. Prove that at a gathering of seventeen people, there must be either (i) three mutual friends, (ii) three mutual enemies, or (iii) three mutual strangers.

Solution:

Let A be one of the 17 people and partition the remaining 16 into 3 sets: E the enemies of A, F the friends of A and U those who are unacquainted with A. The largest of these 3 sets must contain at least 6 people. For definiteness, let us suppose that F contains 6 or more people. If any 2 members of F, say B and C, are friends, then A,B,C are 3 mutual friends. Now suppose no 2 people in F are friends. Choose any B in F and partition the remaining 5 people in F into 2 sets: X the enemies of B and Y those who are unacquainted with B. Either X or Y contains at least 3 people. If it is X and any 2 people in X, say C and D, are enemies, then B,C,D are 3 mutual enemies. Otherwise, no 2 people in X are enemies and then all the people in X are mutual strangers. Similarly if Y contains 3 or more people, we obtain either 3 mutual strangers (B and 2 people in Y), or 3 mutual enemies all in Y. Obviously, the same argument can be repeated if either E or U has 6 members.

440. Let Π be a polygon and let P be any point inside Π. If every line segment joining P to any other point inside or on Π lies completely in Π, we say that Π is visible from P. For example, in Figure 1, the polygon is visible from A but not from B because the line segment BQ passes outside the polygon. Prove that the set of all points from which Π is visible is a convex set. (A set S in the plane is convex if the line segment joining any two points A and B of S lies completely in S — see Figure 2.)

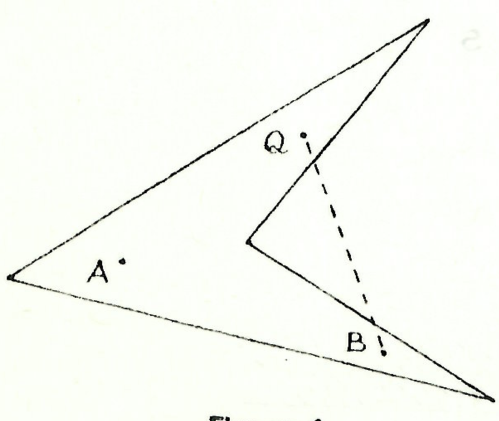


Figure 1

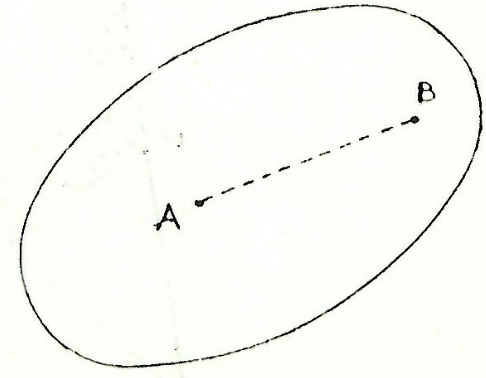


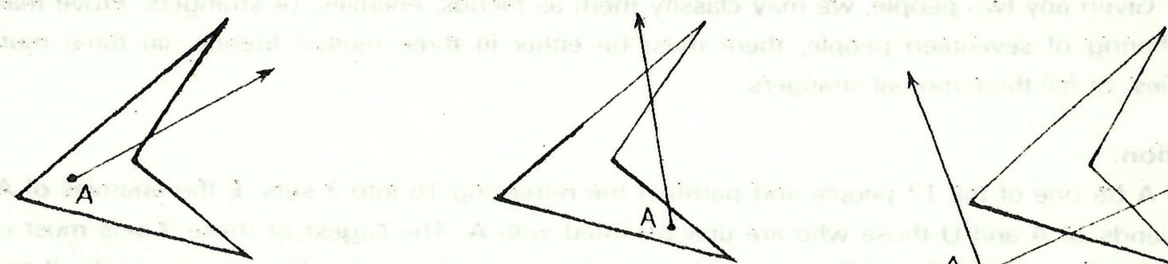
Figure 2

Solution:

Let S be the set of points from which Π is visible.

(i) We show first that if A is a point of S, then any ray from A, that is an infinite straight line segment beginning at A, must meet Π in precisely one point.

If the ray contained more than one point of Π, then the one most remote from A would not be visible from A. If the ray contains no points of Π, then A must be outside Π. Now there are rays through A which meet Π in 2 or more points and some of these points are not visible from A by the previous remark. (See the diagrams, where the parts of Π visible from A are shaded more heavily.)



(ii) Now let A and B be any two points of S. We want to show that the whole line segment joining A and B lies in S.

The ray from B going directly away from A contains just one point of Π, say X. By (i), X is the only point on the ray from A going through B. In particular, there is no point of Π on AB.

Let P be any point on Π. By (i), P is the only point of Π on the rays from A and B going through P. Each ray BT lying in the angle XBP meets Π in a unique point, Y(T) say, and as the direction of BT sweeps out the angle from BX to BP, the points Y(T) form the portion of Π connecting X to P. This portion of Π must lie wholly in the shaded region in the diagram. Each ray AS lying in the angle XAP must intersect this portion of Π and so, by (i), no point of Π lies in the triangle ABP. Thus, if C is any point on AB, then P is visible from C. This holds for any point P on Π, so C is in S. This proves that S is convex.

K. Svendsen (Busby High School) submitted a praiseworthy attempt, but unless I am misinterpreting it, I don't think his argument is quite complete.