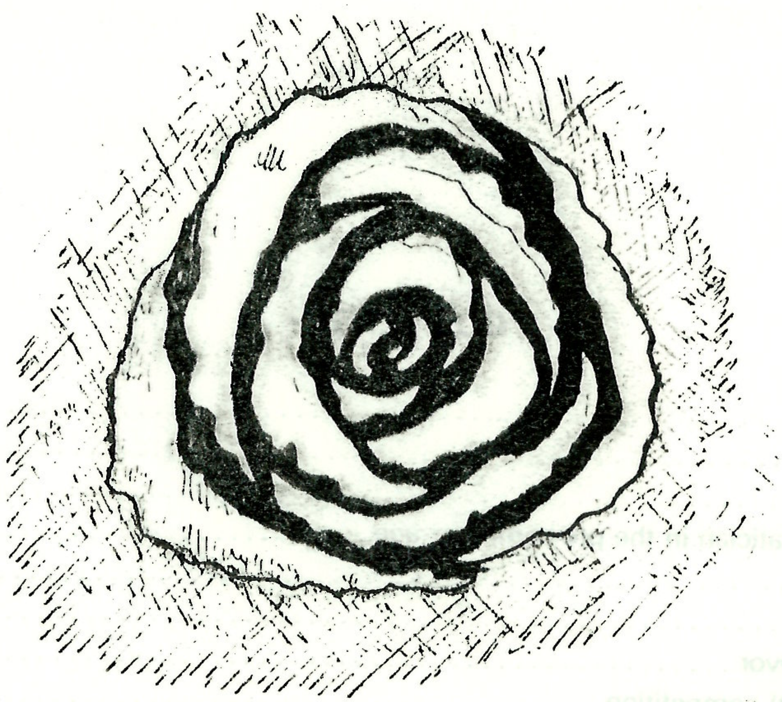
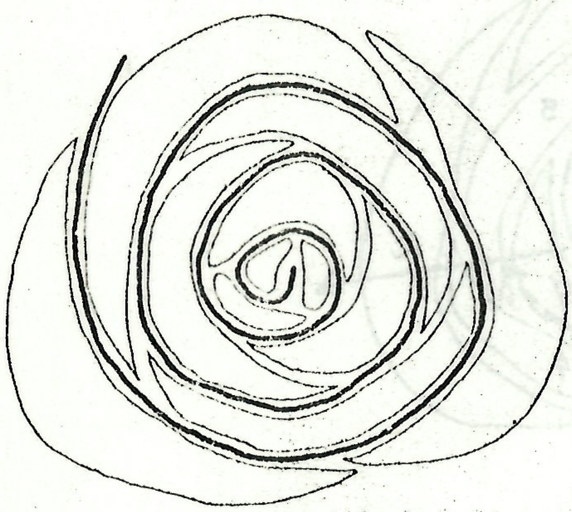
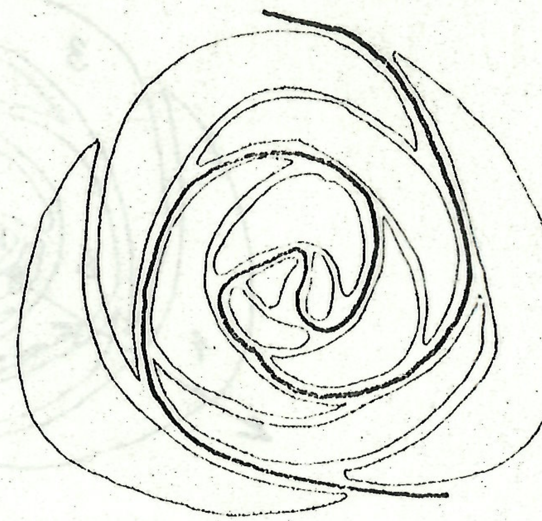
SPIRALS

Summer is here and the school year is almost over! What a glorious time it is to ramble about the bush, the beach, the garden, enjoying the sunshine and nature's abundant beauty. But while we are having fun, we might as well keep an observant eye on our surroundings, keep a sharp look-out not just for the lurking dangers of a trap-door spider or a jelly-fish, but for other less dramatic but equally exciting phenomena of our world. We don't have to travel to exotic lands in search of adventure, indeed something quite wonderful may be waiting for our discerning eye at the "bottom of the garden". Let us start a little investigation with the humble celery. First we cut a bunch about 3-4 inches above the bottom, a little above the solid conical core of the plant, where the stalks are still fairly tightly packed, to get a cross section showing the overlapping growth pattern, as in the picture below.

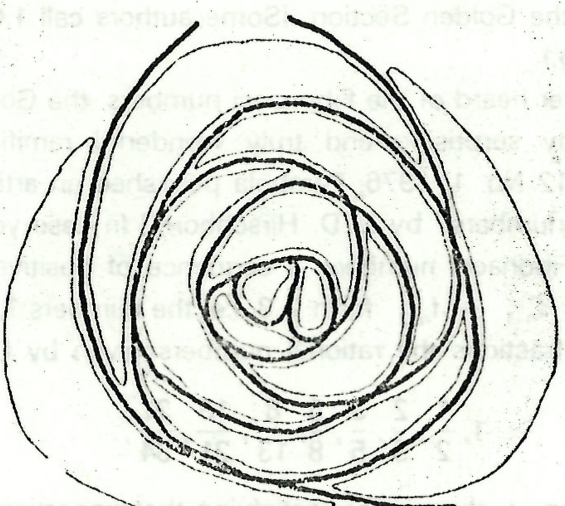


To analyse this "leaf arrangement" (or using the accepted botanical term phillotaxis), of the celery we reproduce the cross section in a simplified diagram, borrowed from the beautiful and instructive "picture book" "Patterns in Nature" by Peter S. Stevens. The whorling pattern of the cross section suggests a spiral path winding through between the gaps of the stalks, but neither the single spiral (shown on the left) nor the double spiral (shown on the right) will traverse all the gaps between the stalks by themselves.

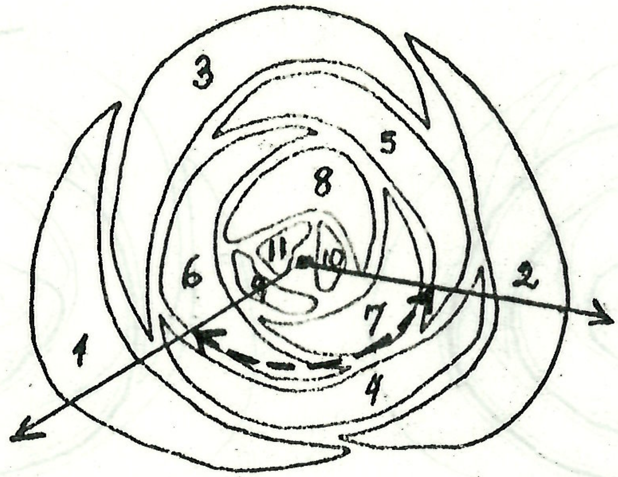




But the two spirals superimposed on each other will outline the stalks of the celery completely, as you can see it in the diagram below.



It is clear that we could have drawn the spirals through the centres of the stalks instead of in the gaps between the stalks. To get a more accurate picture of the distribution of the celery stalks let us look at the cross section again, this time without the spirals, but the stalks numbered in the order of their growth.



We observe that the angle subtended from the centre of the bunch, its arms passing through approximately the mid points of two consecutive stalks in the order of their growth is pretty much the same, as indicated in the diagram. Allowing for irregularities of the plant this angle is very close to 137° 30′ 28″, and is called the *Fibonacci Angle.* In radian measure it is equal to $\frac{1}{2}(3-\sqrt{5})\,2\pi$, and hence it can be expressed in terms of the Golden Section, indeed it is equal to $2\pi\phi^2$, where $\phi = .618\ldots$, the Golden Section. (Some authors call 1.618. . . the Golden Section, clearly the reciprocal of our ϕ.)

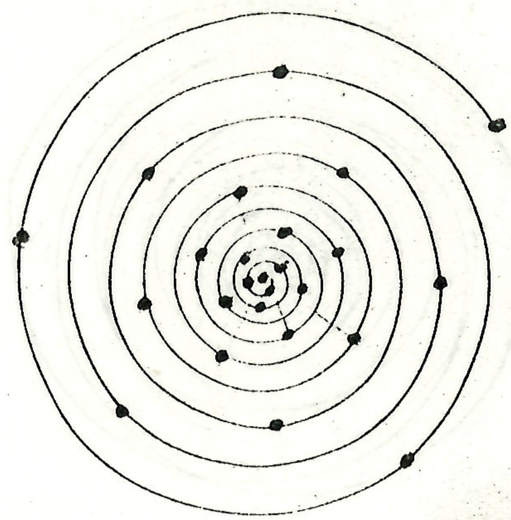
Some of you probably never heard of the Fibonacci numbers, the Golden Ratio (or Section), their connections, and their many surprising and truly wonderful ramifications in nature, art and mathematics. [Note: In Vol. 12 No. 1, 1976, Parabola published an article on "Fibonacci numbers, Pascal's triangle, and prime numbers" by M.D. Hirschhorn.] In case you have no ready access to information, we define the Fibonacci numbers, a sequence of positive integers generated by the rule: $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 3$, i.e. the numbers 1, 1, 2, 3, 5, 8, 13, 21, 34,. . . . Then we call the Fibonacci fractions the rational numbers given by f_n/f_{n+1}, i.e. the sequence of fractions

$$1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \ldots$$

We define the Golden Section, ϕ, the number satisfying the proportion $a/b = (b-a)/a$. From this: setting $b = 1$, solving the ensuing quadratic equation in a, and neglecting the negative root, we obtain $\phi = \frac{1}{2}(\sqrt{5}-1) = 0.618\ldots$. With the use of a pocket calculator you may easily convince yourself that

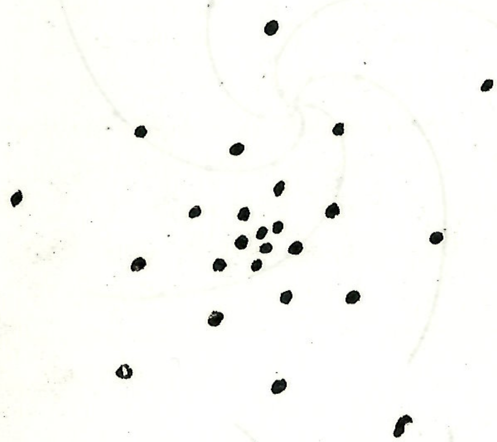
$$\lim_{n\to\infty} f_n/f_{n+1} = \phi.$$

Let us now return to our celery. The spiral patterns and the distribution of the stalks suggests a "mathematical model" in the shape of a logarithmic spiral, with points lying on it so that the circular arc between any two consecutive points is the "Fibonacci Angle", as shown in the diagram.

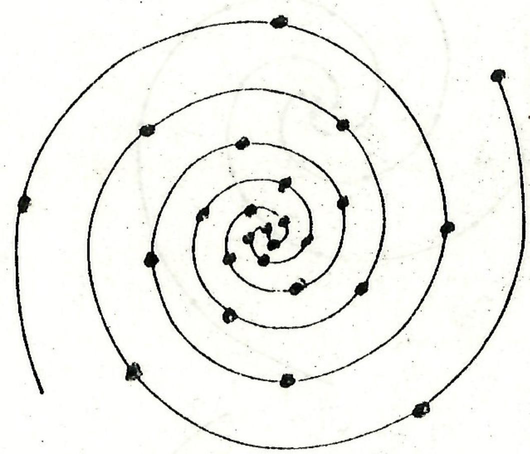


Spiral 1

Retaining the array of points, but erasing Spiral 1, we may trace a double spiral, which we shall call Spiral 2, passing through exactly the same points as shown in the diagrams below.

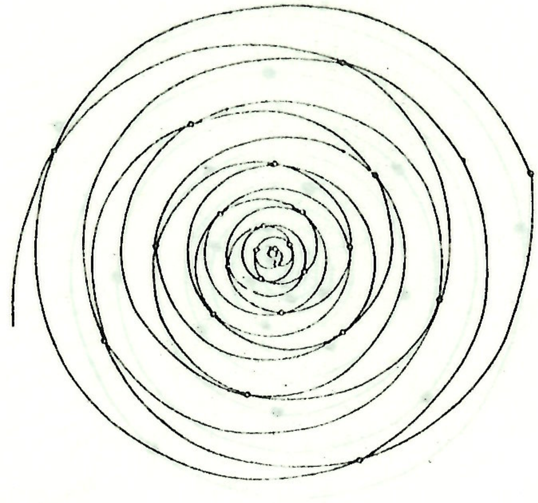


Array of points

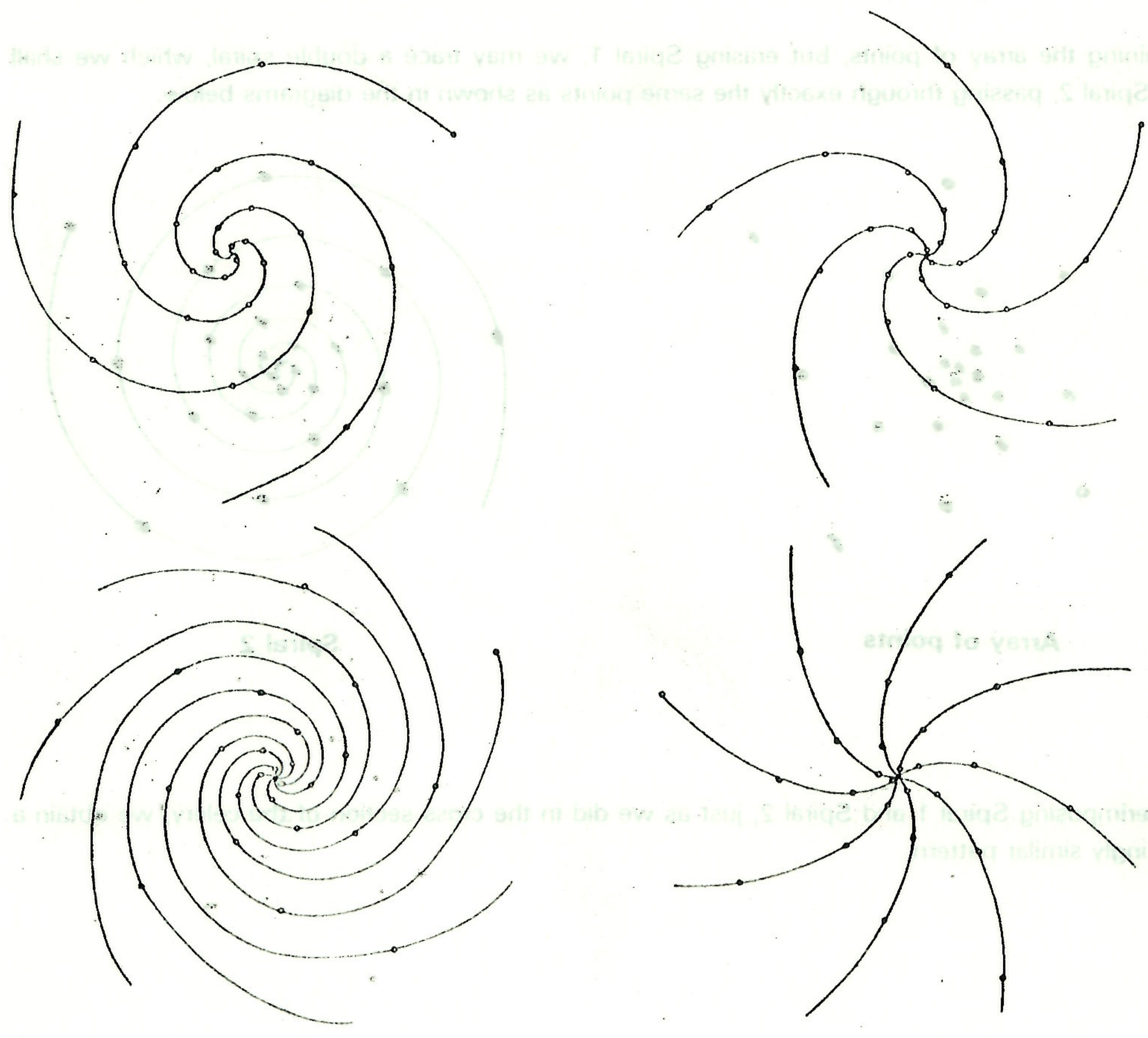


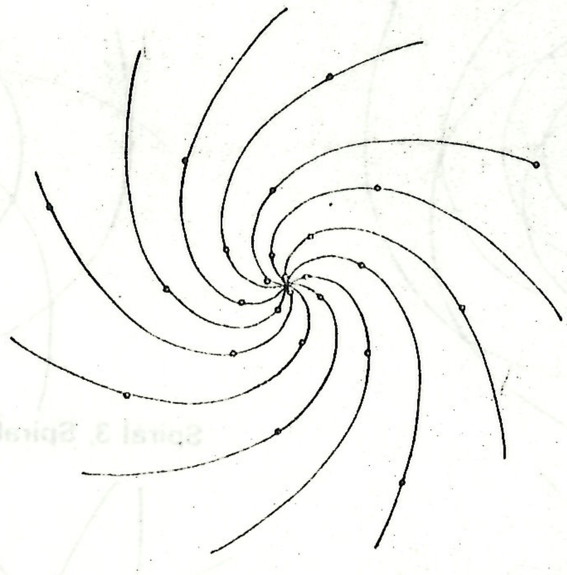
Spiral 2

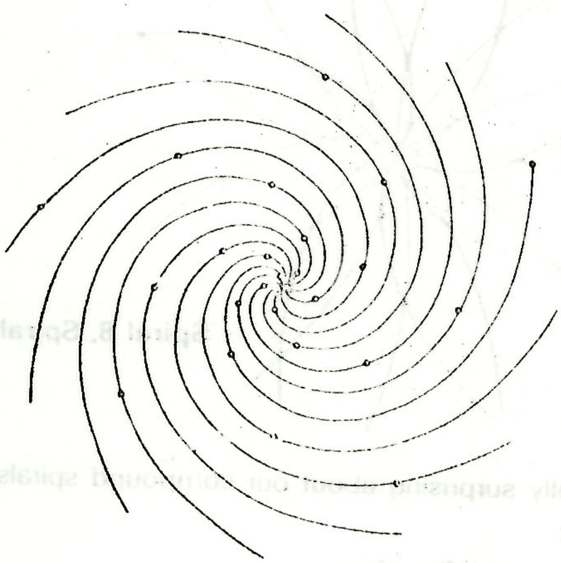
Superimposing Spiral 1 and Spiral 2, just as we did in the cross section of the celery, we obtain a strikingly similar pattern.

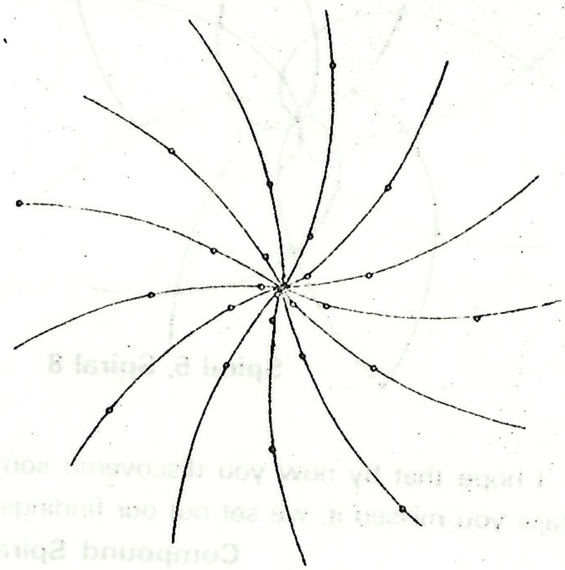


But this is not all. Using exactly the same array of points we can trace other spirals with three, five, etc branches as shown in the following pictures:









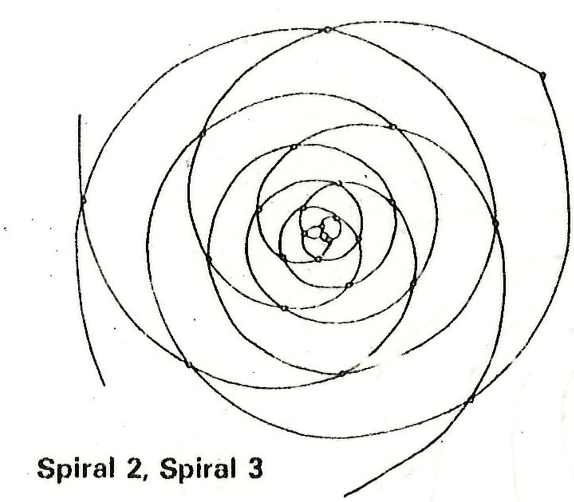
It is interesting to note that we cannot draw regular 4-, 6-, 9- or 10-branched spirals which will pass through the same array of points.

Now, just as we did with Spiral 1 and Spiral 2 to obtain the "celery pattern", we may superimpose some of our new spirals onto each other with the following restrictions:

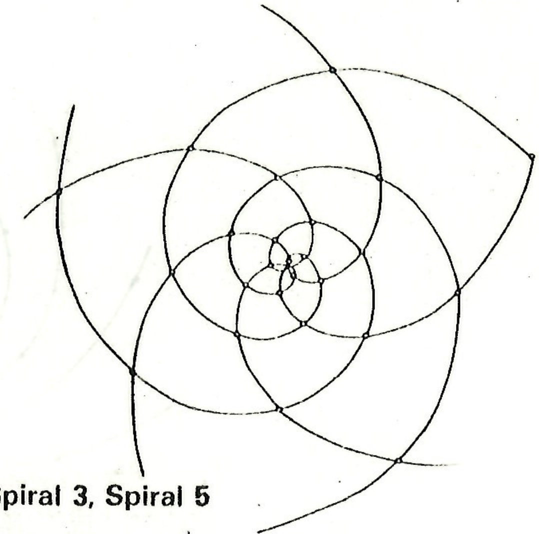
(a) they turn in opposite directions, and

(b) they cross each other only at the points of the original array.

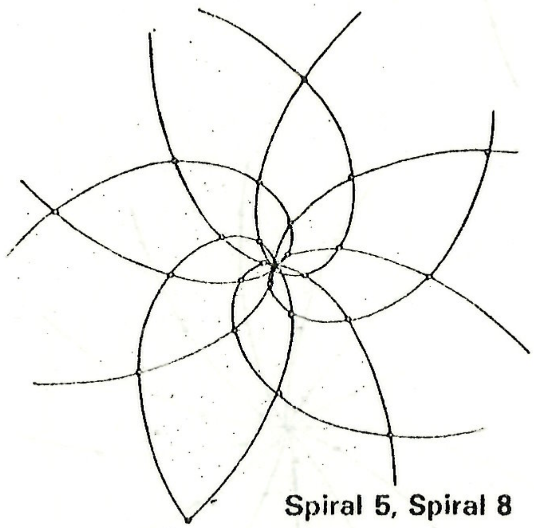
We show here some of the possibilities of the overlapping component spirals with the notation (Spiral k, Spiral ℓ) where k and ℓ represent the number of branches of the spiral.



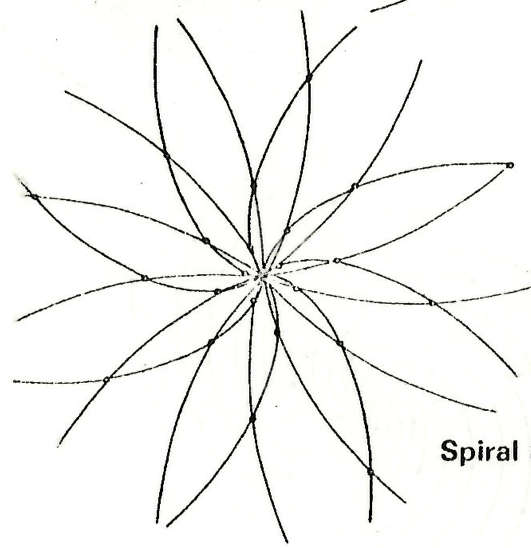
Spiral 2, Spiral 3



Spiral 3, Spiral 5



Spiral 5, Spiral 8



Spiral 8, Spiral 13

I hope that by now you discovered something really surprising about our compound spirals. In case you missed it, we set out our findings in a table:

Compound Spiral	Fibonacci Fraction
(Spiral 1, Spiral 2)	1/2
(Spiral 2, Spiral 3)	2/3
(Spiral 3, Spiral 5)	3/5
(Spiral 5, Spiral 8)	5/8
(Spiral 8, Spiral 13)	8/13

The surprising fact is that our original array of points not only generates *all* Fibonacci fractions, it generates *only* Fibonacci fractions. As we recognised the essential pattern of the celery cross section in our (Spiral 1, Spiral 2) compound, it is not difficult to discern the patterns of other plants in the other compound spirals we generated.

Try to draw some other spirals using the "Fibonacci Angle". You may modify the array of points by using a tighter wound initial spiral. Try to match the possible compound spirals with natural forms.

I would be happy to see your efforts.

A. Nikov