THE ART OF COUNTING. II*

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Naturally, anyone interested in mathematics should be familiar with bridge, so we shall turn our attention to calculating the probabilities of various events at the bridge table. To find the probability of a certain event, we must count the number of favourable outcomes and divide this number by the total number of possible hands. A hand at bridge consists of 13 cards chosen from a deck of 52 cards. How many ways are there of doing this? The first card we pick up can be any of 52, the second can be any of 51, and so on, down to the thirteenth which can be any of 40. But, as in all sensible card games, the order of receiving the cards does not matter, so we divide our number of choices by 13! which is the number of ways of rearranging the 13 cards in the hand. The number of possible bridge hands is therefore

$$C^{52}_{13} = 52.\ 51.\ 50.\ \ldots .41.40/13.12.11.\ \ldots .2.1 \approx 6.4 \times 10^{11},$$

which is fortunately quite a large number. However, you may not be quite convinced by this simple counting argument. After all, at the bridge table, you only get one card in every four. Does your chance of receiving a given hand depend on where the dealer sits? We will return to this problem in a little while.

Conditional probabilities.

Consider a situation with finitely many equally likely outcomes and let A be an event consisting of some of the possible outcomes. The probability that the event A occurs, which we shall denote by p(A), is given by

$$p(A) = \text{number of outcomes in A / total number of outcomes.}$$

To see what this means, consider the probability of drawing an ace from a well-shuffled pack of cards. There are four aces, that is four favourable outcomes, and 52 possible outcomes altogether, so the probability of drawing an ace is 4/52 = 1/13. This should not be too surprising.

Now, let us consider the probability that the events A and B occur in two successive trials. For concreteness, suppose we want the chances of drawing two aces from a well-shuffled pack. At the first draw, there are 4 aces among 52 cards, so the probability of the first ace (event A) is 4/52 = 1/13. But once we have drawn one ace, there remain 3 aces among 51 cards, so the probability of getting a second ace (event B) after the first has been drawn is 3/51 = 1/17. The overall probability of obtaining two aces in succession is (1/13) . (1/17) = 1/221. In general, we have the equation

* *Part I appeared in Parabola, Volume 16, Number 2.*

number of outcomes with events A and B in successive trials

= (number of outcomes in A) . (number of outcomes in B after A has occurred at the first trial).

If we divide both sides by the total number of possible outcomes of the two trials, we get the equation for the probabilities

$$p(A \text{ and } B) = p(A) \,.\, p(B \text{ after A occurs at the first trial})$$

The last probability is usually abbreviated as $p(B|A)$. So we have

$$p(A \text{ and } B) = p(A) \,.\, p(B|A).$$

The same idea extends to any number of events, for example

$$p(A \text{ and } B \text{ and } C) = p(A) \,.\, p(B|A) \,.\, p(C|A \text{ and } B).$$

With this background, let us see whether we can calculate the probability of obtaining a completely specified bridge hand. Suppose, for a start, that you are sitting on the dealer's left, so that you are dealt the first card. The first card dealt must be one of the magic thirteen you require, the next three dealt to the other players must come from the 39 cards you do not require, then the fifth card dealt must be one of your chosen hand, and so on. So your chances of getting this particular hand are

$$\frac{13}{52}\;\frac{39}{51}\;\frac{38}{50}\;\frac{37}{49}\;\frac{12}{48}\;\frac{36}{47}\;\frac{35}{46}\;\frac{34}{45}\;\frac{11}{44}\;\frac{33}{43}\ldots\frac{1}{4}\;\frac{3}{3}\;\frac{2}{2}\;\frac{1}{1} = \frac{13!\;\;39!}{52!}\,.$$

What happens if you are next in line, so that you receive the second card dealt? Now your probability of getting a completely specified hand is

$$\frac{39}{52}\;\frac{38}{51}\;\frac{37}{50}\;\frac{12}{49}\;\frac{36}{48}\;\frac{35}{47}\;\frac{34}{46}\;\frac{11}{45}\;\frac{33}{44}\;\frac{11}{43}\ldots\frac{3}{4}\;\frac{1}{3}\;\frac{2}{2}\;\frac{1}{1} = \frac{13!\;\;39!}{52!}\,.$$

You will now be able to see that the probability of any player getting a completely specified hand is $1 / C^{52}_{13}$. So, if we are considering only one hand, we can indeed forget the peculiarities of the table situation and work in terms of C^{52}_{13} equally likely outcomes. For example, let us find the probability of getting a Yarborough, that is a hand without aces, kings, queens, jacks or tens. To get a Yarborough, we must be dealt 13 cards out of the 32 remaining cards (9's, 8's, 7's . . ., 2's); the number of ways of doing this is C^{32}_{13}. So the probability of a Yarborough is just

$$C^{32}_{13} \,/\, C^{52}_{13} \approx 0.00055.$$

More (difficult) bridge problems

What is the probability that each of the four players receives exactly one ace? The first player must get 1 out of the 4 aces and 12 out of the other 48 cards; this can be done in $C^4_1 \,.\, C^{48}_{12}$ ways out of a total of C^{52}_{13}. Now the second player must get 1 out of the 3 remaining aces and 12 out of the other 36 cards; this can be done in $C^3_1 \,.\, C^{36}_{12}$ ways out of a total of C^{39}_{13}. For the third player, we have 2 aces among 26 cards, so he succeeds in getting his ace in $C^2_1 \,.\, C^{24}_{12}$ cases out of a total of C^{26}_{13}. The remaining cards then include 1 ace and all go to the fourth player. So the probability that each player has exactly one ace is

$$(C^4_1 \, . \, C^{48}_{12} \, / \, C^{52}_{13})(C^3_1 \, . \, C^{36}_{12} \, / \, C^{39}_{13})(C^2_1 \, . \, C^{24}_{12} \, / \, C^{26}_{13}) \approx 0.11.$$

(Note that the probability that you receive exactly one ace in the deal is $C^4_1 \, . \, C^{48}_{12} \, / \, C^{52}_{13} \approx 0.44$, 0.44, but what you hold affects the other players' chances of getting an ace, and this makes the counting a little tricky.) Another way to analyse this question is to consider the cards one by one as they are dealt to the players. First, let us work out the probability that each player receives his ace in the first round of dealing. This is just

$$\frac{4}{52}\,\frac{3}{51}\,\frac{2}{50}\,\frac{1}{49}\,\frac{48}{48}\,\frac{47}{47}\cdots\frac{2}{2}\,\frac{1}{1} = \frac{4!\;48!}{52!}.$$

Each player can receive his ace as any of his 13 cards, which gives 13^4 different combinations. As you can check, any particular one of them (for example, the first player's ace is his seventh card, the second player's is his fourth, the third player's is his second and the fourth player's is his thirteenth) has probability 4! 48!/52!. So the overall probability that each player receives an ace is 13^4.4! 48!/52!. This is the same as before, which is just as well.

Let us pass on to a problem that many a declarer has had to face. The declarer can see 26 cards (his own hand and dummy) and these include 9 hearts. The two highest hearts missing from these are the queen and the 8. The 4 hearts in his opponents' hands could be split 4–0, 3–1, or 2–2. Should the declarer play for a 2–2 split, or try to take a finesse in the hope that a 3–1 split is favourable? First, let us work out the probabilities of the various distributions of hearts among the opponents' hands. For a 4–0 split to occur the hand with the 4 hearts could be constructed in C^{22}_9 ways (since it must contain the 4 hearts and 9 out of the remaining 22 cards), and it could be on either side of the declarer, so a 4–0 split occurs with probability

$$2 \, . \, C^{22}_9 \, / \, C^{26}_{13} = 11/115.$$

For the 2–2 situation, the hand on declarer's left could be constructed in $C^4_2 \, . \, C^{22}_9$ ways (using 2 out of the 4 hearts and 9 out of the other 22 cards) and the other hand contains all the remaining cards, so a 2–2 split occurs with probability

$$C^4_2 \, . \, C^{22}_{11} \, / \, C^{26}_{13} = 468/1150.$$

This leaves 572/1150 as the probability of a 3–1 split. (You could check this independently.) Now let us examine the results of the declarer's possible plays. First, suppose the declarer plays off the ace and the king. One of his opponents will win a trick with the queen if he also holds two small hearts, so this strategy succeeds in the case of a 2–2 split (probability 468/1150) and in the case when one player has a singleton queen of hearts (probability $2 \, . \, C^{22}_{12} \, / \, C^{26}_{13} = 143/1150$); the overall chance of success is 611/1150. The next possible play is to try the king followed by a finesse of a small heart. This loses when the appropriate opponent holds the queen and one, two or three small hearts. The overall probability of this is

$$C^3_1 \, . \, C^{22}_{11} \, / \, C^{26}_{13} + C^3_2 \, . \, C^{22}_{10} \, / \, C^{26}_{13} + C^3_3 \, . \, C^{22}_9 \, / \, C^{26}_{13} = 1007/2300.$$

So the chance of success with this strategy is 1293/2300. If the declarer tries a finesse on the first round, his chance of success is only 1/2. (Can you see why?)

Problems

1. Poker is another popular game among probabilists. How many different poker hands are there? (A poker hand consists of 5 cards dealt from a deck of 52 cards.) What is the probability that a poker hand has no pairs? What is the probability that a poker hand contains four cards of the same value (that is four aces, four twos, etc.)? What is the probability of a straight flush (that is five consecutive cards in the same suit — the ace can count as the lowest card or the highest one)? Does your answer explain why a straight flush beats four of a kind?

2. A bridge player had no ace in three consecutive hands. What is the probability of this occurring? Do you think he should go and buy another lucky charm?

3. A trembling declarer can see 11 hearts in his own hand and dummy; the missing cards are the king and the eight. Should declarer play a finesse of a small heart, or play the ace in the hope that a singleton king will fall?

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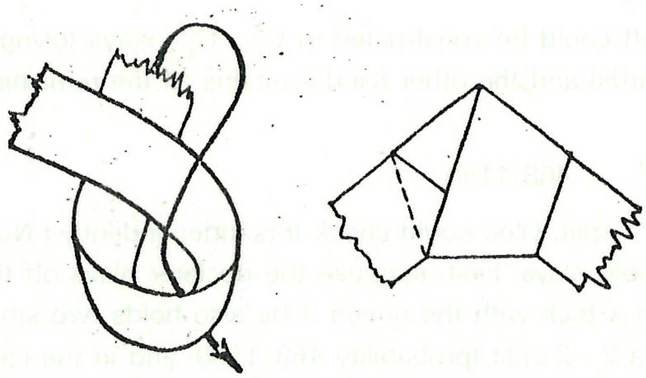
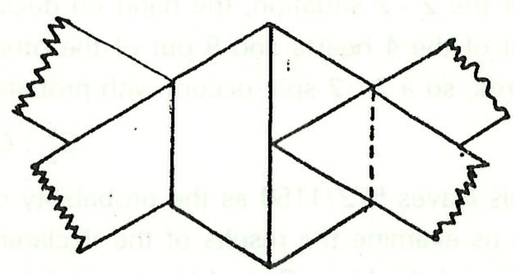
POLY-KNOTS

The ancient Greeks used ruler and compass to construct certain regular polygons. Now, you can tie them in knots. If you tie a hitch in a strip of paper and flatten the folds, you will get a regular pentagon. If you flatten out a reef knot (but not a granny knot), you will get a regular hexagon. Can you prove that the figures really are a regular pentagon and a regular hexagon?



Penta-knot



Hexa-knot