CIRCLE-INK

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A circle in two-dimensions is easily described by its familiar equation: the circle with centre at the point (a,b) and radius r has the equation $(x-a)^2 + (y-b)^2 = r^2$. A circle in three-dimensions is a little trickier; as well as the centre and radius of the circle, we now have to specify the plane in which it lies. The most convenient way to do this is to give a pair of simultaneous equations. For example, the intersection of a plane and a sphere is always a circle; this circle is described by a pair of equations, one being the equation of the plane and the other the equation of the sphere. Let us see how this works.

Example 1. Let C be the circle given by the intersection of the plane P: $2x - y + 1 = 0$ and the sphere S: $x^2 + y^2 + z^2 = 3$. Now S is the sphere with centre (0,0,0) and radius $\sqrt{3}$. The centre of the circle of intersection is the foot of the perpendicular from (0,0,0) to the plane P, namely $(-2/5, 1/5, 0)$ and its radius is $\sqrt{(14/5)}$. (See Figure 1, which shows the view looking down the z axis.)

Example 2. Let C′ be the circle given by the pair of equations P′: $x + z + 1 = 0$ and S′: $x^2 + y^2 + z^2 - 6y - 2z = 0$. Now S′ is the sphere with centre (0,3,1) and radius $\sqrt{10}$, since we can rewrite its equation as $x^2 + (y-3)^2 + (z-1)^2 = 10$. The centre of our circle is $(-1,3,0)$ which is the foot of the perpendicular from the centre of the sphere S′ to the plane P′, and the radius of the circle is $2\sqrt{2}$. Can you verify these facts?

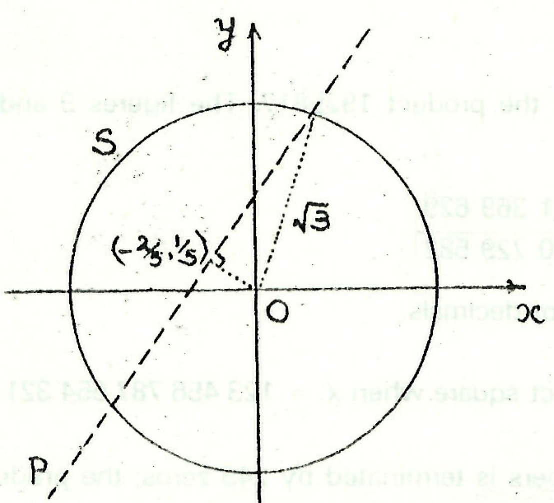

Figure 1

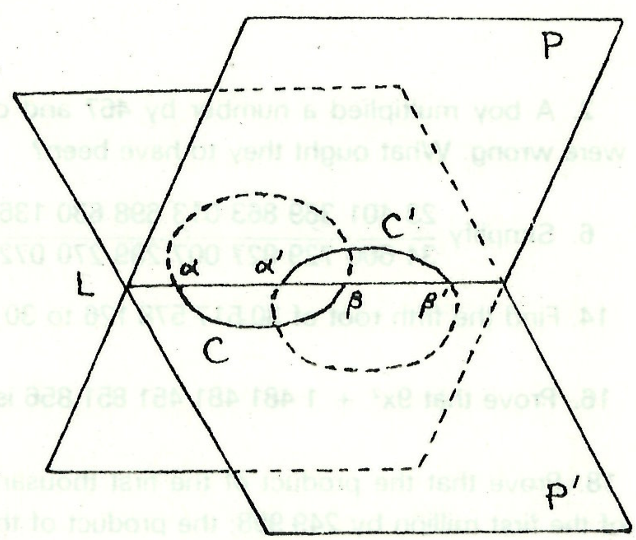

Figure 2

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Now that we know what we are talking about, let us consider something more interesting and three-dimensional. How do we decide whether two circles in space are linked? We shall settle this question for the circles C and C′ specified in examples 1 and 2 above.

Let L be the line of intersection of the two planes $P: 2x - y + 1 = 0$ and $P': x + z + 1 = 0$ which contain our circles. By solving these equations simultaneously, we obtain the parametric equations for this line, namely

$$L: x = t, \ y = 2t+1, \ z = -t-1.$$

Thus to specify a point on L, all we have to do is give the corresponding value of t; as t varies from $-\infty$ to $+\infty$, the point moves along L from one end to the other.

The condition for the two circles to be linked is shown in Figure 2. Both circles must meet the line L and the points of intersection of the circle C with L must be interleaved with the points of intersection of C′ with L, as shown in the Figure. (We shall ignore various special cases where some of these points of intersection coincide. They are degenerate and, anyway, cause no real difficulties.)

The points of intersection of the circle C with the line L are found from the equations

$$S: x^2 + y^2 + z^2 = 3, \quad L: x = t, y = 2t+1, z = -t-1.$$

We obtain a quadratic in t:

$$t^2 + (2t+1)^2 + (-t-1)^2 = 3,$$

that is

$$6t^2 + 6t - 1 = 0.$$

There are two roots for t, say

$$\alpha = -(3+\sqrt{15})/6, \quad \beta = -(3-\sqrt{15})/6,$$

corresponding to two points on L. Similarly, the points of intersection of the circle C′ with L are found from

$$S': x^2 + y^2 + z^2 - 6y - 2z = 0, \quad L: x = t, y = 2t+1, z = -t-1,$$

and again we obtain two values for t, say

$$\alpha' = -1/3, \quad \beta' = 1,$$

and thus two points of intersection on L. Now

$$\alpha < \alpha' < \beta < \beta',$$

so the points of intersection of C and L are interleaved with those of C′ and L. We conclude that the two circles C and C′ are linked.

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A FEW MORE TEASERS FROM OUR ANCIENT DOCUMENT:

Find, by inspection, a factor of 6 000 004 049.

Factorise — (1) 31 249 999 999 999; (2) 31 999 999 999 999 999.