PICTURES OF DIVISION*

Nothing unites the English like war.
Nothing divides them like Picasso.
—H. Mills.

Books on set theory are very keen on Venn diagrams which provide a useful technique for visualising complicated identities involving unions and intersections of sets. In this article, we will see how the same idea can be used to picture relations involving greatest common divisors and least common multiples.

First, a few words about greatest common divisors and least common multiples. Let a and b be two integers. If a number d divides both a and b, we call it a common divisor of a and b. Among all the common divisors of a and b, there is a greatest one which we call the greatest common divisor of a and b. It is denoted by (a,b). If (a,b) = 1, we say that a and b are relatively prime; in this case, the only common divisors of a and b are ± 1. For example, the greatest common divisor of 24 and 56 is 8; the numbers 15 and 22 are relatively prime. A number m is called a common multiple of a and b if it is divisible by both of them. Among all the positive common multiples of a and b, there is a smallest one which we call the least common multiple of a and b. It is denoted by [a,b]. For example, [24,56] = 168 and [15,22] = 330. Now suppose we know the prime factorisations of a and b, say

$$a = p^{\alpha} q^{\beta} r^{\gamma} \ldots, \quad b = p^{\varrho} q^{\sigma} r^{\tau} \ldots, \tag{1}$$

where p,q,r,... are distinct primes and the exponents are non-negative integers. If d divides both a and b, then we can write

$$d = p^{\lambda} q^{\mu} r^{\nu} \ldots$$

where the exponent on each prime is less than or equal to the corresponding exponents occurring in a and b, that is $\lambda \leqslant \alpha$, $\lambda \leqslant \varrho$, and so on. If d is the greatest common divisor of a and b, then the exponent on each prime must be equal to the smaller of the corresponding exponents occurring in a and b, that is $\lambda = \min(\alpha,\varrho)$, and so on. Similarly, if

$$m = p^{\lambda} q^{\mu} r^{\nu} \ldots$$

is a common multiple of a and b, then the exponent on each prime must be greater than or equal to the corresponding exponents occurring in a and b. Therefore, if m is the least common multiple of a and b, the exponent on each prime must be the larger of the corresponding exponents in a and b, that is $\lambda = \max(\alpha,\varrho)$, and so on. Summarising all this, we see that if a and b have the prime factorisations in (1), then

$$(a,b) = p^{\min(\alpha,\varrho)} q^{\min(\beta,\sigma)} r^{\min(\gamma,\tau)} \ldots \quad [a,b] = p^{\max(\alpha,\varrho)} q^{\max(\beta,\sigma)} r^{\max(\gamma,\tau)} \ldots . \tag{2}$$

* *Your Editor would like to thank Herve Nayna for contributing some of the identities in this article and for suggesting several improvements to the original manuscript.*

For example, if $a = 2^6 3^2 5^1 7^0$ and $b = 2^5 3^3 5^0 7^1$, then $(a,b) = 2^5 3^2 5^0 7^0$ and $[a,b] = 2^6 3^3 5^1 7^1$.

Now we are going to represent a and b by the two blobs shown in Figure 1. The elements of the blobs a and b are just the prime divisors of a and b, respectively, repeated according to their multiplicities. For example, if $a = 2^6 3^2 5$, say, the blob for a has the elements 2,2,2,2, 2,2,3,3 and 5. Suppose a and b have the prime factorisations in (1). Then the prime p appears α times in the blob for a and ϱ times in the blob for b. Consequently, the number of times p appears in the intersection of the two blobs is $\min(\alpha,\varrho)$ and the number of times p appears in the union of the two blobs is just $\max(\alpha,\varrho)$. Comparing this with the formulae (2), we see that we can interpret the intersection of the blobs for a and b as the blob for (a,b) and the union of the blobs for a and b as the blob for [a,b]. We can also interpret the product of a and b. From (1),

$$ab = p^{\alpha+\varrho} q^{\beta+\sigma} r^{\gamma+\tau} \dots ;$$

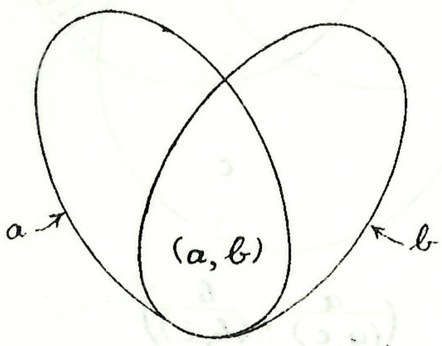

Figure 1

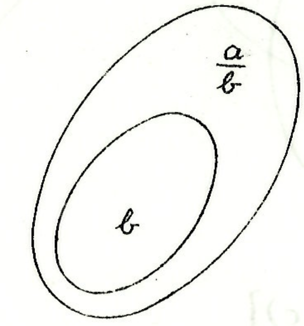

Figure 2

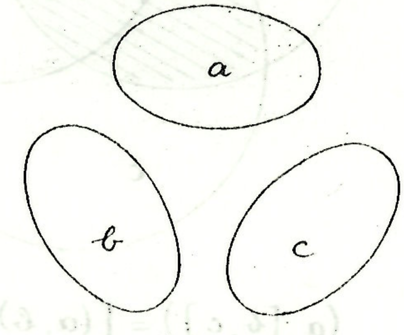

Figure 3

Now, if we "add" the blobs for a and b, counting any elements in the overlap of the two blobs twice, we find that p does indeed occur $\alpha+\varrho$ times, and so on. So we can interpret the sum of the blobs for a and b as the blob for ab. (Do not confuse this sum with the union in which common elements are only counted once.) Division corresponds to subtraction of blobs: if the blob b lies wholly inside the blob a, then b divides a and a/b is represented by the part of the blob a lying outside the blob b. (See Figure 2.) Disjoint blobs correspond to pairwise relatively prime numbers. (See Figure 3.) Note that in this case, the union and the sum of the blobs are the same, that is the least common multiple of the numbers is equal to their product. In other words, if a,b,c,... are relatively prime in pairs, then a number is divisible by each of a,b,c,... if and only if it is divisible by their product abc.... (This is often useful; see, for example, the solution to problem 431 in this issue.)

Armed with this machinery, we can now prove some identities. First,

$$(a,b)[a,b] = ab.$$

We can see this from Figure 1: the blob for the left-side is the sum of the intersection and the union of the blobs for a and b and this is the same as the sum of the blobs for a and b. In particular, $[a,b] = ab$ if $(a,b) = 1$, as we have already mentioned. Next,

$$(a,[b,c]) = [(a,b),(a,c)], \quad [a,(b,c)] = ([a,b],[a,c]). \tag{3}$$

These identities correspond to the distributive laws for unions and intersections:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

We have drawn the first identity in Figure 4. Can you draw the second?

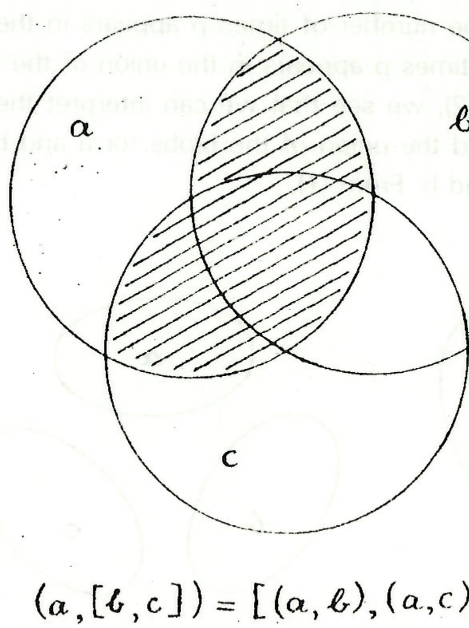

Figure 4

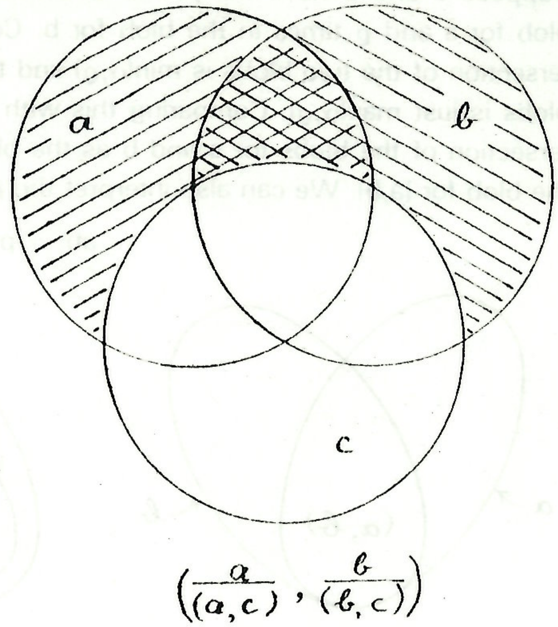

Figure 5

It's just common sense, isn't it? Let us see if we can deal with the problems labelled common sense in Parabola, Volume 16, Number 1. First,

$$(a/(a,c),\ b/(b,c)) = (a,b)/(a,b,c).$$

To obtain the left-side, we take the intersection of the part of a outside c with the part of b outside c, giving the cross-hatched region in Figure 5. This is clearly equal to the part of the intersection of a and b outside c, as required. Next, we shall use Figure 6 to simplify the expression

$$(a(a,c,d)/(a,c)(a,d),\ b(b,c,d)/(b,c)(b,d)).$$

The first diagram in Figure 6 shows the region a(a,c,d)/(a,c)(a,d) and the second diagram shows the other half of our expression. Imagine one diagram on top of the other and you will have the intersection of the two regions. Lo and behold,

$$(a(b,c,d)/(a,c)(a,d),\ b(b,c,d)/(b,c)\ (bd,)) = (a,b)(a,b,c,d)/(a,b,c)(a,b,d).$$

Note that some care is required in drawing the figures in order to make sure that none of the possible regions are accidentally omitted. If the figure has some accidental special feature then it only proves a special case of the general identity. For example, Figure 3 could not be used to prove the identities (3).

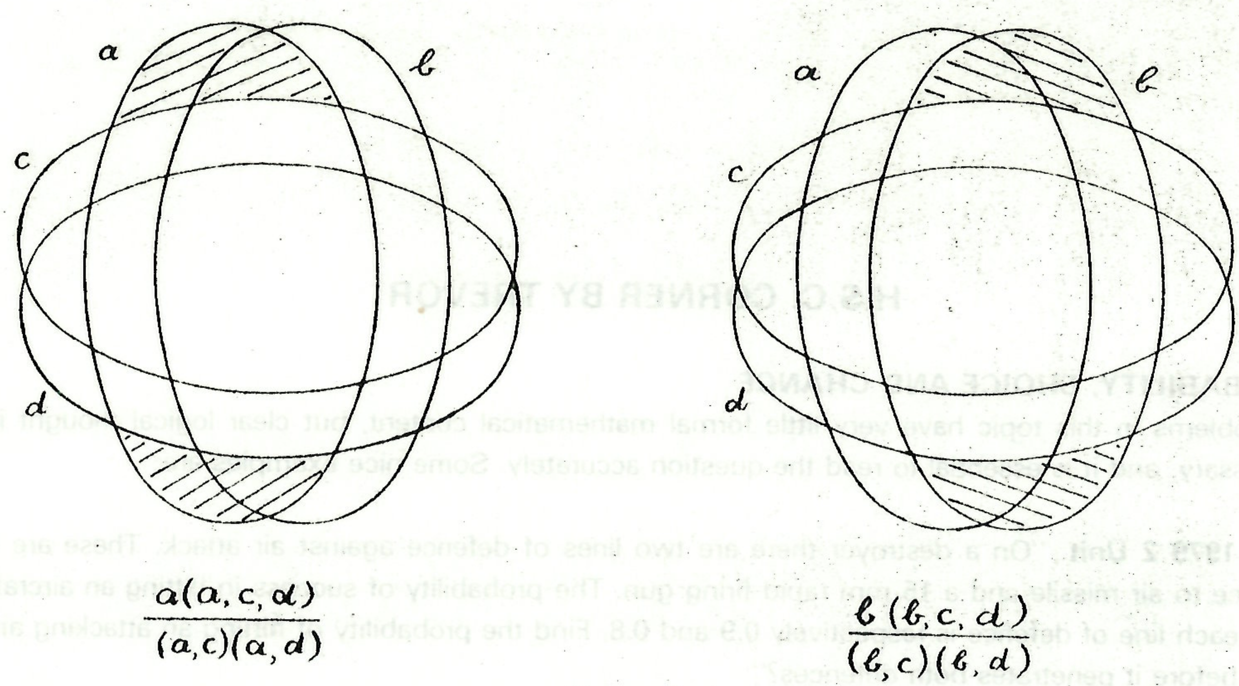

Figure 6

Here are some more problems to test your skill. Prove the identities

$$([a,b],[a,c],[b,c]) = [(a,b),(a,c),(b,c)],$$

$$(ab,cd) = (a,c)(b,d)\,(a/(a,c),d/(b,d))(c/(a,c),b/(b,d)).$$

Translate the first of these into a statement about unions and intersections of sets.

Given that $(a,b,c,d) = 1$, simplify the following expressions:

$$(a,(b,d)(c,d)/(b,c,d)^2),$$

$$(a/(a,b,d)(a,c,d),c/(c,d)),$$

$$(a/(a,d)(a,b,c),\ b/(b,c)(a,b,d)).$$

What happens if $(a,b,c,d) > 1$?

☆ ☆ ☆

A FEW MORE TEASERS FROM OUR ANCIENT DOCUMENT:

Prove that the square of a number consisting of n 6's is found by writing (n − 1) 4's, a 3, (n − 1) 5's and a 6.

Show that the middle digit in the square of the number 12 345 678 987 654 321 is 2.

Prove $\left[\frac{59347}{18162}\right]^3 + \left[\frac{8693}{18162}\right]^3 = 35.$