H.S.C. CORNER BY TREVOR

PROBABILITY, CHOICE AND CHANCE

Problems in this topic have very little formal mathematical content, but clear logical thought is necessary, and it is essential to read the question accurately. Some nice examples are:

80.4 1979 2 Unit. "On a destroyer there are two lines of defence against air attack. These are a surface to air missile and a 15 mm rapid firing gun. The probability of success in hitting an aircraft with each line of defence is respectively 0.9 and 0.8. Find the probability of hitting an attacking aircraft before it penetrates both defences?"

This is a very nice exercise in reading the question and clear thinking. An aircraft will get through if *neither* the gun or the missile hits. The probability of *not hitting* the aircraft is $0.1 \times 0.2 = .02$. Hence the chance of hitting the aircraft before it gets through is $1 - .02 = 0.98$.

80.5 1978 3 Unit. "In a certain strain of plant the probability that a seed will produce a pink flower is 1/4. Determine the least number of seeds that must be planted in order that the probability of obtaining at least one pink flower exceeds 0.99".

This question is really quite hard. First assume that n seeds are planted, and *none* produce a pink flower. The probability of this is $(3/4)^n$, and n must be sufficiently large that $(3/4)^n < 1 - 0.99 = 0.01$. This is because the probability of producing *at least one* pink flower is complementary to producing none. Taking logs to the base 10,

$$n \log_{10}(.75) < -2, \quad \text{i.e.} \ -0.1249n < -2$$

$$\text{Therefore } n > 2/0.1249 = 16.01$$

Thus at least 17 flowers must be planted.

80.6 The following is another nice example: "Given an unlimited number of red, white and blue flags, what is the probability that in 8 flags there is at least one of each colour?"

This is another nice example of complementary probability: Since the number of flags is unlimited, the total number of ways of choosing the flags is 3^8. The number of ways of choosing *no red flag* is 2^8. Similarly the number of ways of choosing no blue flag and no white flag is 2^8 each. But remember that the possibility of all red flags, all blue flags or all white flags are each counted twice. Thus the required probability of at least one of each colour is

$$\frac{3^8 - (3.2^8 - 3)}{3^8} = 1 - \frac{85}{729} = \frac{644}{729}$$

Now try the following two problems.

80.7 In problem 80.4, the destroyer carries 2 missiles and 2 guns. Two enemy planes attack simultaneously. How should the captain deploy his defences to minimise the chance of penetration? What is the best chance of surviving the attack?

80.8 In a certain class there are 7 boys and 6 girls. A committee of 7 persons is formed. What is the probability that the girls have a majority of members of the committee?

(Answers on page 24)

ANSWERS TO QUESTIONS IN VOL. 16 No. 2)

Problem 80.3 (continued) Here are the solutions to the problems posed in Volume 16, Number 2. (The numbers (1), (2) and (3) refer to equations in the earlier article.)

(i) (a) From the first example discussed, we know that $\alpha^2 + \beta^2 + \gamma^2 = -2q$ and hence

$$\alpha^2 + \beta^2 = -2q - \gamma^2,\ \beta^2 + \gamma^2 = -2q - \alpha^2,\ \gamma^2 + \alpha^2 = -2q - \beta^2.$$

We therefore want the equation with roots $-2q-\alpha^2$, $-2a-\beta^2$ and $-2-\gamma^2$. Put $z = -2q-x$, so that $x = -2q-z$, and substitute in (3). This gives the required equation

$$(z+2q)^3 - 2q(z+2q)^2 + q^2(z+2q) + r^2 = 0.$$

Alternatively, put $y = -2q-x^2$, so that $x^2 = -2q-y$ and $x(y+2q) - qx - r = 0$ from the given equation (1). If we substitute $x = r/(y+q)$ in (1) and multiply through by $(y+q)^3/r$, we get the equation

$$(y+q)^3 + q(y+q)^2 + r^2 = 0.$$

Check that this is the same equation as before.

(b) Let the required equation be

$$x^3 + Ax^2 + Bx + C = (x-1-1/\alpha)(x-1-1/\beta)(x-1-1/\gamma).$$

Then, using (2),

$$A = -3 - 1/\alpha - 1/\beta - 1/\gamma = -3 - (\beta\gamma + \gamma\alpha - \alpha\beta)/\alpha\beta\gamma = -3 + q/r$$

$$B = (1 + 1/\alpha)(1 + 1/\beta) + (1 + 1/\beta)(1 + 1/\gamma) + (1 + 1/\gamma)(1 + 1/\alpha)$$

$$= 3 + 2(1/\alpha + 1/\beta + 1/\gamma) + 1/\alpha\beta + 1/\beta\gamma + 1/\gamma\alpha$$

$$= 3 - 2q/r + (\alpha + \beta + \gamma)/\alpha\beta\gamma = 3 - 2q/r.$$

and $C = 1\ (1 + 1/\alpha)(1 + 1/\beta)(1 + 1/\gamma)$

$$= -1 - (1/\alpha + 1/\beta + 1/\gamma) - (1/\alpha\beta + 1/\beta\gamma + 1/\gamma\alpha) - 1/\alpha\beta\gamma$$

$$= -1 + q/r + 1/r.$$

So the required equation is

$$x^3 - (3 - q/r)x^2 + (3 - 2q/r)x - (1 - q/r - q/r) = 0.$$

Alternatively, let $y = 1 + 1/x$, so that $x = 1/(y-1)$. If we substitute in (1) and multiply through by $(y-1)^3$, we get the same equation in the form

$$r(y-1)^3 + q(y-1)^2 + 1 = 0.$$

(ii) Suppose a, b and c are the roots of $x^3 + qx + r = 0$. Then, as we have shown in the preceding examples,

$$a + b + c = 0$$

$$a^3 + b^3 + c^3 = -3r$$

and

$$a^5 + b^5 + c^5 = 5qr.$$

If we take $r = -1$ and $q = -2$, the cubic is

$$x^3 - 2x - 1 = (x+1)(x^2 - x - 1) = 0$$

and it has the roots -1, $\frac{1}{2}(1+\sqrt{5})$ and $\frac{1}{2}(1-\sqrt{5})$. This gives us six sets of solutions for the given system of equations, namely

$$(a,b,c) = (-1, \tfrac{1}{2}(1 \pm \sqrt{5}), \tfrac{1}{2}(1 \pm \sqrt{5})), (\tfrac{1}{2}(1 \pm \sqrt{5}), -1, \tfrac{1}{2}(1 \pm \sqrt{5})), (\tfrac{1}{2}(1 \pm \sqrt{5}, \tfrac{1}{2}(1 \pm \sqrt{5}), -1).$$

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A NOTE ON AN H.S.C. QUESTION

Problem 80.1 of Volume 16 No. 1 discussed one way of solving the equation

$$|x + 1| > |x - 1|.$$

One of the easiest ways to solve equations of this type is to remember that $|x| = \sqrt{x^2}$. Thus, since $|x-1|$ and $|x+1|$ are both positive, we can square both sides of the above inequality to get

$$(x+1)^2 > (x-1)^2$$

$$x^2 + 2x + 1 > x^2 - 2x + 1$$

$$4x > 0$$

$$x > 0.$$

Using this method, it is easy to show that $|x+2| > |x-1|$ if and only if $x > -\frac{1}{2}$. Similarly $|+2| > 8\,|x-1|$ if and only if $63x^2 - 132x + 60 < 0$

i.e.

$$21x^2 - 44x + 20 < 0$$

So x lies between the roots of $21x^2 - 44x + 20$

i.e.

$$2/3 < x < 10/7$$

Ramses Youhana, North Sydney Boys High School sent in a good solution.