PROBLEM SECTION

You are warmly invited to submit solutions to one or more of the problems that follow. Please begin each problem on a new page and make sure that each answer bears your name, year and school. Solutions together with the names of successful solvers will be published in the issue after next.

To spur on your efforts, remember that the best and most consistent solvers this year will receive book prizes. If you found the International Olympiad problems 450-454 too much of a handful, don't be totally discouraged. They are designed to test the most brilliant students from countries with a long tradition of school problem solver (and this tradition leads to constant practice, which certainly improves standards in this as in other worthwhile pursuits). To be confronted with such problems is a little akin to finding oneself playing a tennis match against Bjorn Borg. Even to win a single point would be an achievement to be treasured, and the experience, though perhaps a ittle humiliating, would be considered an honour by many. At any rate that is my excuse for again selecting several problems which have been used in National Olympiads, or in preliminaries to Olympiads. The first six have been used in such competitions in Sweden; the next five in Britain this year. The last problem is submitted by P. Rider, who has discovered this property of the Pascal triangle.

Happy puzzling!

467. In a plane are 127 toothed cog-wheels, numbered (1), (2), . . ., (127). The teeth of wheel (1) engage those of wheel (2), and similarly (2) is engaged with (3), (3) with (4), and finally (127) with (1). Can the cogwheels of the system be turned? Justify your assertion.

468. Find a rapid way of calculating $(999{,}999{,}999)^3$; and perform the calculation.

469. Solve the following equation

$$\cos x + \cos^5 x + \cos 7x = 3.$$

470. In a circle centre O are inscribed a regular polygon of n sides, and a regular polygon of (n + 1) sides. Show that it is possible to choose a vertex P of the first polygon and a vertex Q of the second such that the angle POQ is less than $\dfrac{360}{n(n+1)}$ degrees.

471. A triangle lies inside a polygon. Prove rigorously that the perimeter of the triangle is less than the perimeter of the polygon.

472. Show that any three real numbers x, y, z which satisfies

$$\sin x + \sin y + \sin z = 0$$
$$\cos x + \cos y + \cos z = 0$$

also satisfy

$$\sin 2x + \sin 2y + \sin 2z = 0$$
$$\cos 2x + \cos 2y + \cos 2z = 0 .$$

473. Prove that the equation $x^n + y^n = z^n$ (n an integer > 1) has no solution in integers x,y,z with $0 < x \leqslant n$, $0 < y \leqslant n$.

474. Find a set S of 7 consecutive positive integers for which there exists a polynomial P(x) of the fifth degree with the following properties:

a) all the coefficients of P(x) are integers

b) P(n) = n for five members, n, of S, including the least and greatest

c) P(n) = 0 for one member of S.

475. On the diameter AB bounding a semicircular region, there are two points P and Q, and on the semicircular arc there are two points R and S, such that PQRS is a square. C is a point on the semicircular arc such that the areas of the triangle ABC and the square PQRS are equal. Prove that a straight line passing through one of the points R and S, and through one of A and B, cuts a side of the square at the in-centre of the triangle.

476. A list of real numbers $a_0, a_1, a_2, \ldots, a_n, \ldots$ has the following properties:

a) $a_{n+1} = 2^n - 3a_n \quad (n \geqslant 0)$

b) $a_{n+1} > a_n \quad (n \geqslant 0)$.

Prove that there is just one such list, and find a_0.

477. In a party of ten persons, among any three persons there are at least two who do not know each other. Prove that in the party there are four persons none of whom knows another of the four.

478. Decide whether the following statement is true or false:– There exists a number N such that every natural number greater than N is the sum of two fourth powers of integers i.e. if Z > N then there exists integers x,y such that $x^4 + y^4 = Z$.

479. Decide whether the following statement is true or false:–
There exists numbers $a_1, a_2, \ldots, a_n$ such that

$$a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx > 0 \quad \text{for all } x.$$

480. In the Pascal triangle the nth row consists of the integers 1, $C_1^n, C_2^n, \ldots, C_n^n$. Let P_n denote their products. Prove that

$$\frac{P_n}{P_{n-1}} = \frac{n^{n-1}}{(n-1)!}.$$