SOLUTIONS TO PROBLEMS FROM VOLUME 16, NUMBER 1

441. Prove that the number 1111...11, consisting of 91 ones, is a composite number.

Solution.

One factorisation is 1111111 × 1000000100000010...10000011 where the second number has 13 ones, each two of which are separated by a block of 6 zeros. There is another similar factorisation in which the first factor is a block of 13 ones and the second has 7 ones separated by blocks of 12 zeros.

Correct solutions were received from J. Cranford (North Sydney Boys High School), N. Brown (Watson High School), K. Lim (St. Ignatius College), D. Everett (Kogarah High School).

442. Factorise the polynomial $x^8 + x^4 + 1$ into factors of at most the second degree.

Solution.

$$\begin{aligned} x^8 + x^4 + 1 &= (x^4 + 1)^2 - (x^2)^2 = (x^4 + 1 + x^2)(x^4 + 1 - x^2) \\ &= [(x^2 + 1)^2 - x^2][(x^2 + 1)^2 - (\sqrt{3}x)^2] \\ &= (x^2 + 1 + x)(x^2 + 1 - x)(x^2 + 1 + \sqrt{3}x)(x^2 + 1 - \sqrt{3}x) \end{aligned}$$

Correct solutions were received from R. Youhana (North Sydney Boys' High School) and K. Lim (St. Ignatius College).

443. None of the numbers a, b or c is zero and each is a root of the equation $x^3 - ax^2 + bx - c = 0$. Find a, b and c.

Solution.

We must find non-zero a, b, c satisfying simultaneously

$$0 = a^3 - a.a^2 + b.a - c = ba - c \qquad (1)$$

$$0 = b^3 - ab^2 + b.b - c \qquad (2)$$

and

$$0 = c^3 - ac^2 + bc - c \qquad (3)$$

Substituting c = ab from (1) into (2) and (3), and cancelling non-zero factors gives

$$0 = b^3 - ab^2 + b^2 - ab = (b + 1)(b - a) \qquad (4)$$

Therefore $b + 1 = 0$, or $b - a = 0$

and

$$0 = a^2b^2 - a^2b + b - 1 = (a^2b + 1)(b - 1) \qquad (5)$$

Therefore $a^2b + 1 = 0$, or $b - 1 = 0$.

Case I. $b + 1 = 0$ and $a^2b + 1 = 0$ yields (a,b,c) = (1,−1,−1) or (−1,−1,1).

Case II. $b + 1 = 0$ and $b - 1 = 0$ yields no solution.

Case III. $b - a = 0$ and $a^2b + 1 = 0$ yields (a,b,c) = (−1,−1,1).

Case IV. $b - a = 0$ and $b - 1 = 0$ yields (a,b,c) = (1,1,1).

Thus there are 3 real solutions (a,b,c) = (1,1,1); (−1,−1,1); or (1,−1,−1).

(Case III also yields the solution $a = b = \varrho$, $c = \varrho^2$ where ϱ is either complex cube root of -1; i.e. $\frac{1}{2}\varrho(1 \pm i\sqrt{3})$.)

Correct solutions were received from J. Taylor (Woy Woy High School) and N. Brown (Watson High School).

444. Prove that if the sum of the positive numbers a, b and c is equal to 1, then $a^{-1} + b^{-1} + c^{-1} \geqslant 9$.

Solution.

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc + ca + ab}{abc} = \frac{(bc + ca + ab)(a + b + c)}{abc} \quad \text{since } a + b + c = 1$$

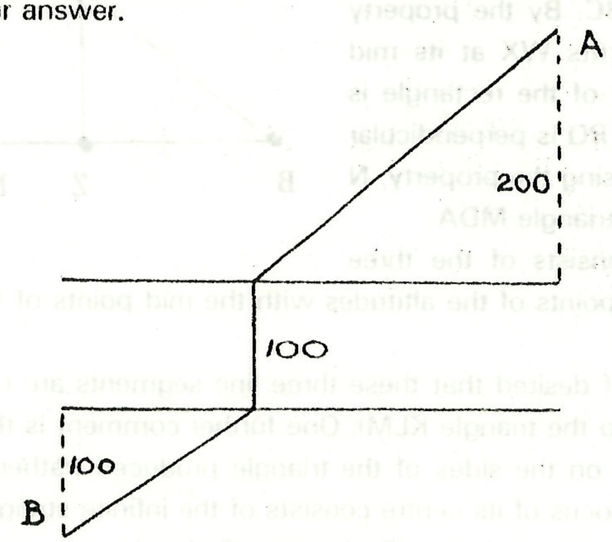
$$= \frac{a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) + 3abc}{abc}$$

$$= \frac{a(b^2 - 2bc + c^2) + b(c^2 - 2ac + a^2) + c(a^2 - 2ab + b^2) + 9abc}{abc}$$

$$= \frac{a(b - c)^2 + b(c - a)^2 + c(a - b)^2}{abc} + 9$$

$\geqslant 9$ since a,b,c > 0 and perfect squares are $\leqslant 0$.
Equality occurs only if $a = b = c = 1/3$.

445. A river 100m wide runs due east-west. Points A and B are on opposite sides of the river and at respective distances of 200m and 100m from its banks. B is 400m further west than A. A road and bridge joining A and B is to be constructed subject to the condition that the bridge must cross the river perpendicularly. What is the shortest possible total length of road and bridge which will join the points? Prove your answer.



Solution.

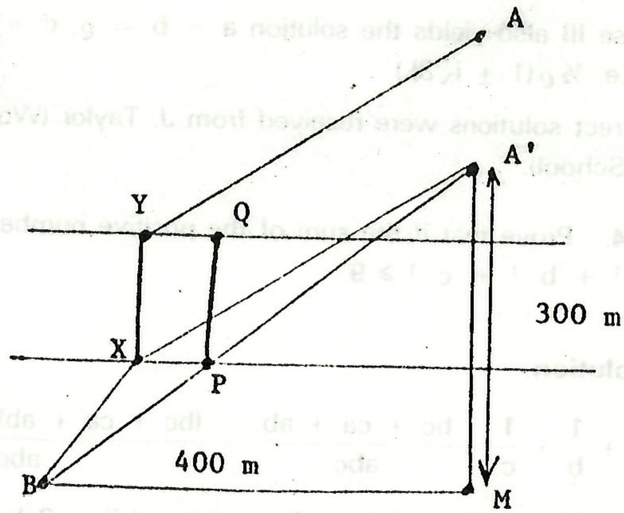
Let A′ be 100 metres due south of A. Let XY represent any bridge position. Then since XY is equal and parellel to A′A, XA′AY is a parallelogram, and *YA = *XA′.

The total road length

*BX + *YA = *BX + *XB′ ⩾ *BA′.

Hence the road length is kept to a minimum by building the bridge at PQ, where BPA′ is straight, and the total length of bridge and road is then

*PA + *BA′ = 100m + $\sqrt{(400^2 + 300^2)}$m
= 600m.



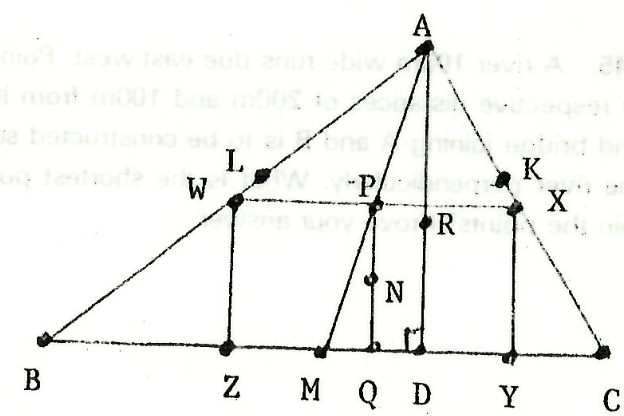
Correct solutions were received from D. Everett (Kogarah High School), K. Lim (St. Ignatius College), N. Brown (Watson High School) and J. Crawford, R. Youhana, C. Ven, A. Jenkinson (North Sydney Boys' High School).

446. A rectangle is drawn so that its four vertices lie on the perimeter of a given acute-angled triangle. Find the locus of the centre of the rectangle as it moves subject to these constraints.

Solution.

N. Brown (Watson High School) submitted an excellent solution using only the easily proved property that a median of a triangle bisects all line segments parallel to the base with end points on the other two sides; thus:-

Label the triangle ABC and first consider all rectangles like WXYZ with two vertices on BC (see figure). Let AD be the perpendicular, and AM the median, from A to BC. By the property stated, since WX||BC, AM cuts WX at its mid point P. Therefore the centre of the rectangle is the mid point N of PQ where PQ is perpendicular to ZY. Note PQ||AD. Again using the property, N lies on the median NR of the triangle MDA.



Thus the desired locus consists of the three line segment joining the mid points of the altitudes with the mid points of the corresponding sides of the triangle.

Comments: One can prove if desired that these three line segments are concurrent (for example, by applying Ceva's theorem to the triangle KLM). One further comment is that if the vertices of the rectangle are permitted to lie on the sides of the triangle produced (rather than, as given, on the perimeter of the triangle) the locus of its centre consists of the infinite straight lines RM, TK, SL.

Correct solution also received from K. Lim (St. Ignatius College).

447. (i) If AB and CD are line segments of equal length ℓ which do not intersect, prove that at least one of AC, AD, BC and BD has a length greater than ℓ.

(ii) Let S be a set of n distinct points in the plane. Consider the $\frac{1}{2}n(n-1)$ line segments connecting all possible pairs of points of S. Any one of the longest of these line segments is called a diameter of S. There may be several diameters; for example, if S consists of the four points at the vertices of a rectangle, both diagonals are diameters. However, by (i), any two diameters must have a point in common.

Prove that a set of n points has at most n diameters.

(iii) Show that for any $n \geqslant 3$, there exists a set of n points having exactly n diameters.

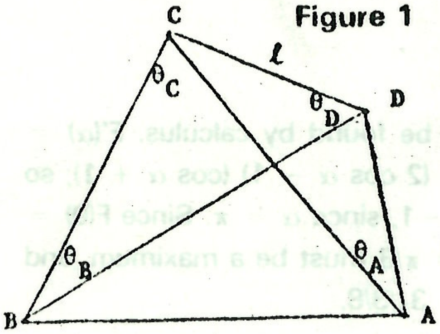
Solution.

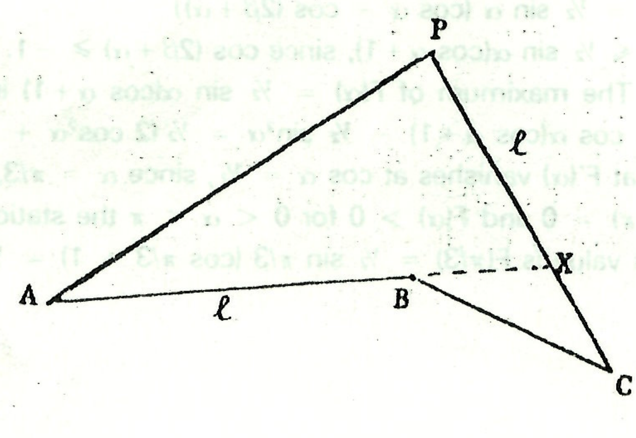
(i) **Case 1.** ABCD is a convex quadrilateral. Re-label the vertices if necessary so that of the four angles θ_A, θ_B, θ_C, θ_D subtended at a vertex by the opposite side of length ℓ, the smallest of θ_A. Then in the triangle ACD, angle D $> \theta_D \geqslant \theta_A$ (see figure 1). Therefore AC* > DC* $= \ell$.

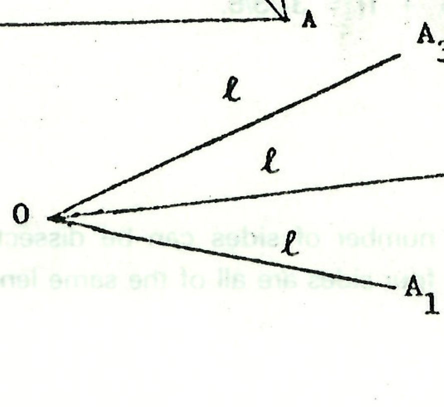
Case 2. (i) ABCD is not convex. Re-label the vertices if necessary so that AB produced cuts CD, at X say. One of the angles CXA and DXA is at least a right angle, thus it is the largest angle in the triangles AXC or AXD respectively. Hence at least one of AC and AD exceeds AX in length, and therefore certainly is longer than AB.

(ii) Suppose there exists a set of n points having (n + 1) diameters. There are 2n + 2 points at the ends of these (n + 1) diameters; hence at least one of the n points, 0 say, is an end point of three or more diameters OA, OA_2, OA_3, say. (See figure. Note that $A_3DA_1 \leqslant 60°$, since $A_1A^*_3 \leqslant \ell$.) Observe that A_2 cannot be the end point of another diameter since it would be impossible for such to intersect both OA_3 and OA_1, as required by (i). Hence if the point A_2 and the line segment OA_2 are deleted we are left with a set of n − 1 points having n diameters. Repeating this argument we would eventually reduce down to a set of 3 points having 4 diameters, an impossibility since there are altogether only $C_2^3 = 3$ line segments connecting the 3 points.

Figure 1

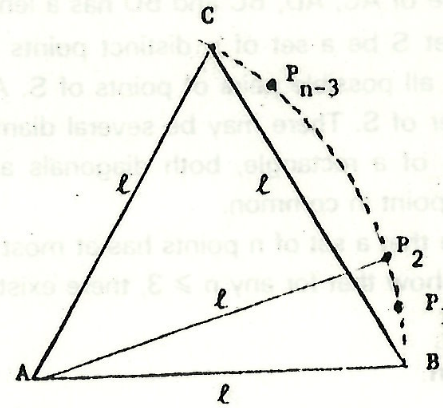






(iii) Let the triangle ABC be equilateral of side length ℓ, and let $P_1, P_2, \ldots, P_{n-3}$ be any $(n-3)$ points on the arc BC of the circle centre A, radius ℓ. There are n diameters (AB, AC, BC, and AP_i, $v = 1, 2, \ldots, n-3$) and n points.

A correct solution was received from K. Lim (St. Ignatius College).

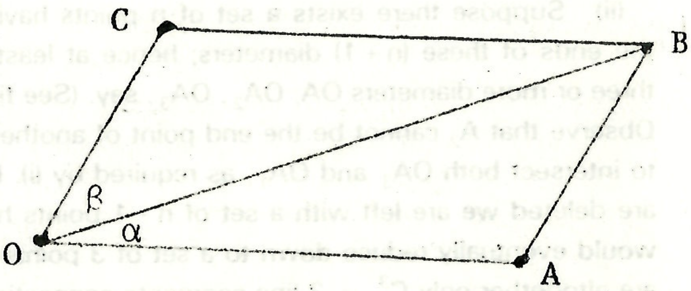


448. Let S be the area of the parallelogram OABC. Prove that $S^3 \leqslant (3\sqrt{3}/8)\, OA^2 \cdot OB^2 \cdot OC^2$.

Solution.

The perpendicular distances between OA and BC is $h = OB \sin\alpha = OC \sin(\alpha + \beta)$. $S = OA.OB \sin\alpha = OA.OC \sin(\alpha + \beta)$. Since $S = OC.OB \sin\beta$ since $OB \sin\beta$ is the perpendicular distance between AB and OC. Therefore $S^3 = OA^2\, OB^2\, OC^2 \sin\alpha \sin\beta \sin(\alpha+\beta)$. It is thus sufficient to show that, for any two positive angles whose sum is less than two right angles,

$$\sin\alpha \sin\beta \sin(\alpha+\beta) \leqslant 3\sqrt{3}/8$$



Using $2\sin\theta\sin\phi = \cos(\theta-\phi) - \cos(\theta+\phi)$ we have

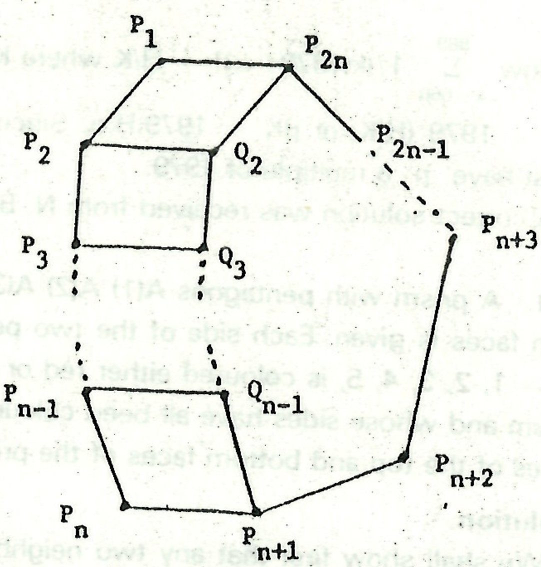
$$\sin\alpha \sin\beta \sin(\alpha+\beta)$$
$$= \tfrac{1}{2}\sin\alpha(\cos\alpha - \cos(2\beta+\alpha))$$
$$\leqslant \tfrac{1}{2}\sin\alpha(\cos\alpha + 1), \text{ since } \cos(2\beta+\alpha) \geqslant -1.$$

The maximum of $F(\alpha) = \tfrac{1}{2}\sin\alpha(\cos\alpha + 1)$ in $0 \leqslant \alpha \leqslant \pi$ can be found by calculus. $F'(\alpha) = \tfrac{1}{2}\cos\alpha(\cos\alpha+1) - \tfrac{1}{2}\sin^2\alpha = \tfrac{1}{2}(2\cos^2\alpha + \cos\alpha - 1) = \tfrac{1}{2}(2\cos\alpha - 1)(\cos\alpha + 1)$; so that $F'(\alpha)$ vanishes at $\cos\alpha = \tfrac{1}{2}$, since $\alpha = \pi/3$, and at $\cos\alpha = -1$, since $\alpha = \pi$. Since $F(0) = F(\pi) = 0$ and $F(\alpha) > 0$ for $0 < \alpha < \pi$ the stationary point at $\alpha = \pi/3$ must be a maximum, and its value is $F(\pi/3) = \tfrac{1}{2}\sin\pi/3\,(\cos\pi/3 + 1) = \tfrac{1}{2}\sqrt{}/2\,(\tfrac{1}{2} + 1) = 3\sqrt{3}/8$.

449. Prove that every regular polygon having an even number of sides can be dissected into lozenges. (A lozenge, or rhombus, is a quadrilateral whose four sides are all of the same length.)

Solution.

Both K. Lim (St. Ignatius College) and N. Brown (Watson High School) submitted the following argument which I have pleasure in substituting for my own rather more complicated solution. I have not used the exact wording of either.



Let $P_1P_2 \ldots P_nP_{n+1} \ldots P_m$ be any convex polygon having all 2n sides of equal length, ℓ, whose opposite sides are parallel e.g. $P_1P_n \parallel P_nP_{n+1}$ etc.

Construct line segments $P_2Q_2, P_3Q_3, \ldots, P_{n-1}Q_{n-1}$ all of length ℓ and parallel to P_1P_{2n}. Join $P_{2n}Q_2, Q_2Q_3, \ldots, Q_{n-2}Q_{n-1}, Q_{n-1}P_{n+1}$ (see figure 1). Clearly the original polygon has been dissected into rhombuses $P_1P_2Q_2P_m$, $P_2P_3Q_3Q_2, \ldots, P_{n-1}P_nP_{n+1}Q_{n-1}$, together with a polygon $Q_2Q_3, \ldots, Q_{n-1}P_{n+1}P_{n+2}, \ldots, P_m$ of 2n − 2 sides.

Since this polygon has the same properties as the one we started with (all sides of length ℓ, opposite sides parallel; e.g. $P_{2n}Q_2 \parallel P_1P_2 \parallel P_{n+1}P_{n+2}$, and convex as its interior angles being equal to or less than a corresponding interior angle of the original, cannot be re-intrant) the same procedure can be repeated, and as the number of sides of the leftover pieces always decreases by two, eventually a dissection into rhombuses results.

450. Let p and q be integers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that p is divisible by 1979.

Solution.

$$\frac{p}{q} = 1 + \frac{1-2}{2} + \frac{1}{3} + \frac{1-2}{4} + \ldots + \frac{1-2}{1318} + \frac{1}{1319}$$

$$= (1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{1319}) - (\frac{2}{2} + \frac{2}{4} + \ldots + \frac{2}{1318}) = \sum_{n=1}^{1319} 1/n - \sum_{n=1}^{659} 1/n$$

$$= \frac{1}{660} + \frac{1}{661} + \ldots + \frac{1}{1318} + \frac{1}{1319}$$

$$= (\frac{1}{660} + \frac{1}{1319}) + (\frac{1}{661} + \frac{1}{1318}) + \ldots + (\frac{1}{k} + \frac{1}{1979-k}) + \ldots + (\frac{1}{989} + \frac{1}{990})$$

$$= \frac{1979}{660.1319} + \frac{1979}{661.1318} + \ldots + \frac{1979}{989.990} \Bigg| = 1979 \sum_{k=660}^{989} 1/(k(1979-k))$$

Now $\sum_{k=660}^{989} 1/(k(1979-k)) = H/K$ where $K = 660.661. \ldots . 1319$ and H is some integer, so that

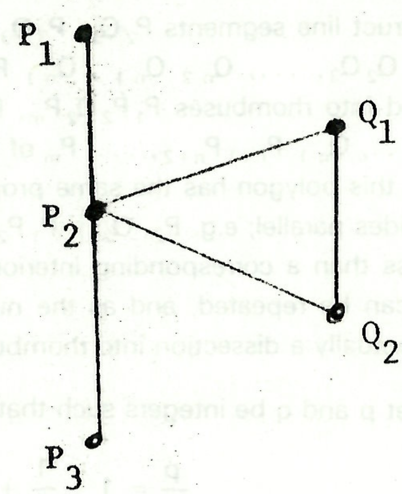
$p/q = 1979\,(H/K)$ or $pK = 1979.H.q$. Since 1979 is a prime number which does not divide K we must have p a multiple of 1979.

A correct solution was received from N. Brown (Watson High School).

451. A prism with pentagons A(1) A(2) A(3) A(4) A(5) and B(1) B(2) B(3) B(4) B(5) as top and bottom faces is given. Each side of the two pentagons and each of the line segments A(i) B(j), for all i,j = 1, 2, 3, 4, 5, is coloured either red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been coloured has two sides of different colours. Show that all ten sides of the top and bottom faces of the prism are the same colour.

Solution.

We shall show first that any two neighbouring edges at the top (or the bottom) must have the same colour. Let $P_1 P_2 P_3$ be three neighbouring vertices. Of the five lines from the middle one, P_2 to the vertices at the opposite end of the prism these must be at least 3 of one colour (say, green) and therefore two neighbouring ones, P_2Q_1, P_2Q_2 both green. Then Q_1Q_2 must be red.



If P_1P_2 were green then from $\Delta P_1P_2Q_1$ we would have P_1Q_1 red, and from $\Delta P_1P_2Q_2$ we would have P_1Q_2 red. But then $\Delta P_1Q_2Q_1$ would have all sides red, which is forbidden. Hence P_1P_2 must be red and similarly P_2P_3 must be the same colour. Since we have now proved that any two neighbouring edges at either end must have the same colour, and Q_1Q_2, P_1P_2, P_2P_3 are red, it follows that all edges at both pentagonal ends must be red.

452. Given a plane, a point P in the plane and a point Q not in the plane, find all points R in the plane such that the ratio (QP + PR)/QR is a maximum.

Solution.

In Figure 1 over page, M is the foot of the perpendicular from Q to the given plane, and R_1, R are the two points in the plane equidistant from M, R lying on PM produced.

Then QR* = QR*$_1$ (from congruent right angled triangles QMR and QMR$_1$) and PR* = PM* + MR*$_1$ > PR*$_1$. Therefore (PQ* + PR*)/QR* > (PQ* + PQ*$_1$)/QR*$_1$. Thus the desired point R must lie on the line PM produced.

In Figure 2 over page, MP is produced to X so that PX* = PQ*. If R is any point on PM produced,

$$(PQ^* + PR^*)/QR^* = XR^*/QR^* = \sin\angle XQR / \sin\angle QXP$$

The maximum of value of this is $1/\sin\angle QXP$, achieved when $\angle XQR = 90°$, i.e. when the semicircle on diameter XR passes through Q. Then, since PQ* = PX*, P is the centre of the semicircle so PR* = PQ*. Hence R must be the point on PM produced such that PR* = PQ*.

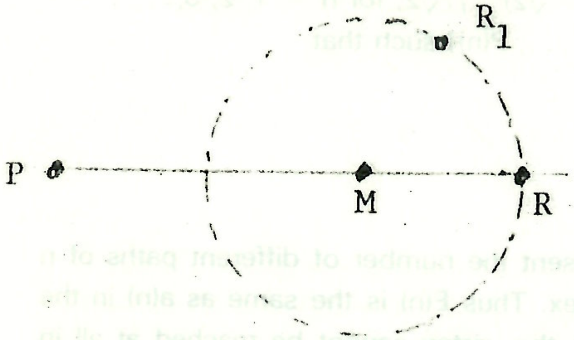
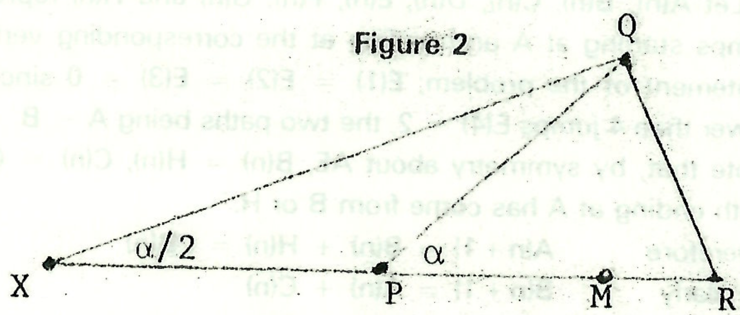


Figure 1

Figure 2



453. Find all real numbers a for which there exist non-negative real numbers x(1), x(2), x(3), x(4) and x(5) satisfying the equations

$$\sum_{k=1}^{5} kx(k) = a, \quad \sum_{k=1}^{5} k^3x(k) = a^2, \quad \sum_{k=1}^{5} k^5x(k) = a^3.$$

Solution.

Let x(1), x(2), x(3), x(4), x(5) be non-negative numbers satisfying the given equations. Then

$$\left(\sum_{i=1}^{5} i\, x(i)\right)\left(\sum_{j=1}^{5} j^5\, x(j)\right) = a.a^3 = (a^2)^2 = \left(\sum_{k=1}^{5} k^3\, x(k)\right)^2.$$

Multiplying out and collecting terms yields

$$\sum_{1\leqslant i\leqslant j\leqslant 5} (ij^5 - 2i^3j^3 + i^5j)\, x(i)\, x(j) = 0.$$

i.e. $\sum_{1\leqslant i\leqslant j\leqslant 5} ij(j^2 - i^2)^2\, x(i)\, x(j) = 0.$

Since all terms on the L.H.S. are $\geqslant 0$, for equality we must have every term equal to 0. Hence at most one of x(1), x(2), x(3), x(4), x(5) is different from 0. If every x(k) = 0, a = 0. If x(i) ≠ 0, the equations become $ix(i) = a$, $i^3x(i) = a^2$, and $i^5x(i) = a^3$ which have the solution x(i) = i when $a = i^2$. Thus only six values of a yield non negative solutions of the equations. viz. a = 0, $a = 1^2$, $a = 3^2$, $a = 4^2$ and $a = 5^2$.

454. Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a(n) be the number of distinct paths of exactly n jumps ending at E.

Prove that $a(2n-1) = 0$ and $a(2n) = \{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}\}/\sqrt{2}$, for $n = 1, 2, 3, \ldots$.
(A path of n jumps is a sequence of vertices (P(0), P(1), ..., P(n)) such that
(i) P(0) = A, P(n) = E,
(ii) $P(i) \neq E$ for $0 \leq i \leq n-1$, and
(iii) P(i) and P(i+1) are adjacent for $0 \leq i \leq n-1$.)

Solution.

Let A(n), B(n), C(n), D(n), E(n), F(n), G(n) and H(n) represent the number of different paths of n jumps starting at A and ending at the corresponding vertex. Thus E(n) is the same as a(n) in the statement of the problem; E(1) = E(2) = E(3) = 0 since the vertex cannot be reached at all in fewer than 4 jumps E(4) = 2, the two paths being A – B – C – D – E and A – H – G – F – E. Note that, by symmetry about AE, B(n) = H(n), C(n) = G(n) and D(n) = F(n). The last jump in a path ending at A has come from B or H.

Therefore $A(n+1) = B(n) + H(n) = 2B(n)$ (1)

Similarly $B(n+1) = A(n) + C(n)$ (2)

$C(n+1) = B(n) + D(n)$ (3)

$D(n+1) = C(n)$ (the frog never returns from E to d) (4)

and $E(n+1) = D(n) + F(n) = 2D(n)$ (5)

These equations (1), (2), (3), (4) and (5) apply for all $n > 0$.

Eliminating A(n), B(n), C(n) and D(n) yields without much difficulty

$$E(n+4) = 4E(n+2) + 2E(n) = 0 \quad \text{for all } n > 0 \qquad (6)$$

Together with E(1) = E(2) = E(3) = 0, E(4) = 1, this recurrence relation determines E(n) for all positive n. Since E(1) = E(3) = 0 the equation gives E(5) = 0 then E(7) = 0; in fact E(2k – 1) = 0 for every natural number k. Again E(6) = 4E(2) = 8, E(8) = 4E(6) + 2E(4) = 36 etc.

Set $K(2n) = [(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}]/\sqrt{2}$, for $n = 1, 2, 3, \ldots$.

Note that $K(2) = [(2+\sqrt{2})^0 - (2-\sqrt{2})^0]/\sqrt{2} = 0\ E = (2)$

and $K(4) = [(2+\sqrt{2})^1 - (2-\sqrt{2})^1]/\sqrt{2} = 2 = E(4).$ (7)

Also $4K(2n+2) - 2K(2n) = (4[(2+\sqrt{2})^n - (2-\sqrt{2})^n] - 2(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}])/\sqrt{2}$

$= ((2+\sqrt{2})^{n-1}\ (4(2+\sqrt{2}) - (2-\sqrt{2})^{n-1}\ (4(2-\sqrt{2}) - 2))/\sqrt{2}$

$= [(2+\sqrt{2})^{n-1}\ (6+4\sqrt{2}) - (2-\sqrt{2})^{n-1}\ (6-4\sqrt{2})]/\sqrt{2}$

$= [(2+\sqrt{2})^{n-1}\ (2+\sqrt{2})^2 - (2-\sqrt{2})^{n-1}\ (2-\sqrt{2})^2]/\sqrt{2}$

$= [(2+\sqrt{2})^{n+1} - (2-\sqrt{2})^{n+1}]/\sqrt{2} = K(2n+4)$

Therefore $K(2n+4) = 4K(m+2) + 2K(2n) = 0.$ (8)

Thus K(2n) satisfies the same recurrence relation as E(2n) (cf (8) and (6)) as well as the same initial values (7). Hence E(2n) = K(2n) for all $n > 0$.

A correct solution was received from N. Brown (Watson High School).