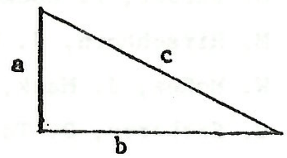
PYTHAGOREAN TRIANGLES

We received an interesting letter from one of our readers, S.J. Cohen, who has found a way of generating sequences of Pythagorean triangles by means of certain irrational square roots.



$$a^2 + b^2 = c^2$$

For example, $\sqrt{2}$ leads to an infinite family of Pythagorean triangles in which the two sides a and b differ by exactly 1. Put $\alpha = \sqrt{2}$. Then $\alpha^2 - 1^2 = (\alpha - 1)(\alpha + 1) = 1$, so

$$\alpha - 1 = \frac{1}{\alpha + 1} = \frac{1}{2 + (\alpha - 1)}.$$

We can substitute for $\alpha - 1$ on the right using this equation to get

$$\alpha - 1 = \cfrac{1}{2 + \cfrac{1}{2 + (\alpha - 1)}},$$

and then substitute for $\alpha - 1$ again to get

$$\alpha - 1 = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + (\alpha - 1)}}},$$

and so on ad infinitum. This yields the continued fraction

$$\alpha - 1 = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

If we truncate this continued fraction after n layers, we obtain an approximation to $\alpha - 1$. The first five approximations are

$$\frac{1}{2};\quad \cfrac{1}{2 + \cfrac{1}{2}} = \frac{2}{5};\quad \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} = \frac{5}{12};\quad \frac{12}{29},\ \text{and}\ \frac{29}{70}.$$

In general, if $\frac{p_n}{q_n}$ is the n-th approximation, then we get the $(n + 1)$-st approximation $\frac{p_{n+1}}{q_{n+1}}$ by adding an extra layer, so

$$\frac{p_{n+1}}{q_{n+1}} = \cfrac{1}{2 + \cfrac{p_n}{q_n}} = \frac{q_n}{2q_n + p_n},$$

which makes it easy to calculate the approximations as far as may be required. Now, we can form Pythagorean triangles by taking

$$a = 2p_n q_n, \quad b = q_n^2 - p_n^2, \quad c = q_n^2 + p_n^2.$$

The first few are given in the table below,

n	p_n	q_n	a	b	c
1	1	2	4	3	5
2	2	5	20	21	29
3	5	12	120	119	169
4	12	29	696	697	985
5	29	70	4060	4059	5741
6	70	169	23660	23661	33461 .

As asserted, the sides a and b satisfy $a - b = \pm 1$.

In the same way starting with $\sqrt{5}$, we get the continued fraction

$$\sqrt{5} - 2 = \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \cdots}}}$$

and this leads to a family of Pythagorean triangles with $2a - b = \pm 1$. The first one is the familiar triangle with $a = 8$, $b = 15$ and $c = 17$.

We can also find a continued fraction expansion for $\sqrt{3}$ with only a small change in the scheme above. If $\beta = \sqrt{3}$, then $\beta^2 - 2^2 = (\beta - 2)(\beta + 2) = -1$, so

$$\beta - 2 = -\frac{1}{2 + \beta} = -\frac{1}{4 + (\beta - 2)},$$

and by repeated substitution, this gives

$$\beta - 2 = -\cfrac{1}{4 - \cfrac{1}{4 - \cfrac{1}{4 - \cfrac{1}{4 - \cdots}}}}$$

If we call the n-th approximation $-\frac{p_n}{q_n}$, then the (n + 1)-st one is

$$-\frac{p_{n+1}}{q_{n+1}} = -\frac{1}{4 - \frac{p_n}{q_n}}, \quad \text{i.e.} \quad \frac{p_{n+1}}{q_{n+1}} = \frac{q_n}{4q_n - p_n}.$$

The Pythagorean triangles with $a = 2p_n q_n$, $b = q_n^2 - p_n^2$ and $c = q_n^2 + p_n^2$ satisfy $c = 2a + 1$.

Mr. Cohen leaves it as a challenge to readers to find the properties of the Pythagorean triangles arising from $\sqrt{t^2+1}$ and $\sqrt{t^2-1}$. Can you deduce from which irrational square root the Pythagorean triangle with $a = 420$, $b = 1189$ and $c = 1261$ is derived?

Mr. Cohen's letter touches on many interesting matters. The following comments may help you to see some of the connections between them.

It has been known since Pythagoras himself that $\sqrt{2}$ is irrational. To the Pythagoreans, this was a terrible shock. It implies that in a 45° right triangle (with $a = b$), the hypotenuse and the side are incommensurable (since $\frac{c}{a} = \sqrt{2}$ is irrational). This largely undermined the Pythagorean philosophy which tried to explain the world by means of positive integers. However, even if we cannot find a 45° right triangle with integer sides, we can still try to approximate it by looking for a right triangle with integer sides having $b = a \pm 1$. Now all Pythagorean triangles whose sides a,b,c have no common factors are given by

$$a = 2pq, \quad b = q^2 - p^2, \quad c = q^2 + p^2.$$

The condition $b = a \pm 1$ becomes $q^2 - p^2 = 2pq \pm 1$, that is

$$(p+q)^2 - 2q^2 = \pm 1.$$

If we can find positive integers p and q satisfying this last equation, we will get the near-isosceles right triangles that we want. More generally to produce a Pythagorean triangle in which $b = ta \pm 1$, we need to find positive integers p and q satisfying

$$(p+tq)^2 - (t^2+1)q^2 = \pm 1.$$

The equation

$$x^2 - Ny^2 = \pm 1,$$

with $N > 1$ and not a square, is usually called Pell's equation. This is not a particularly useful name because, apparently, Pell had nothing to do with the equation. Some people are just lucky. The solutions of Pell's equation are intimately related to the rational approximations to $\sqrt{N}$. Indeed, the equation gives $\frac{x^2}{y^2} - N = (\frac{x}{y} - \sqrt{N})(\frac{x}{y} + \sqrt{N}) = \pm\frac{1}{y^2}$, so

$$\left|\frac{x}{y} - \sqrt{N}\right| = \frac{1}{y^2(\frac{x}{y} + \sqrt{N})},$$

and this is very small if y is large. For example, the large solution

$$70226^2 - 3.40545^2 = 1$$

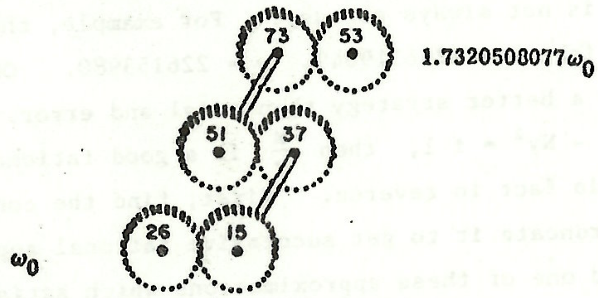
for $N = 3$ gives

$$\frac{70226}{40545} = 1.7320508077 \ldots$$

which agrees with $\sqrt{3}$ to ten decimal spaces. By factoring the numerator and denominator of this fraction,

$$\frac{70226}{40545} = \frac{26}{15} \cdot \frac{37}{51} \cdot \frac{73}{53},$$

we can obtain convenient gear ratios to approximate $\sqrt{3}$:



Let us now return to the particular equation

$$x^2 - (t^2 + 1)y^2 = \pm 1.$$

There is one obvious solution, namely $x_1 = t$, $y_1 = 1$. From this, we can construct infinitely many solutions x_n, y_n by means of

$$x_n + y_n\sqrt{t^2 + 1} = (x_1 + y_1\sqrt{t^2 + 1})^n, \quad n = 2, 3, \ldots$$

For example, $x_2 + y_2\sqrt{t^2 + 1} = x_1^2 + (t^2 + 1)y_1^2 + 2x_1y_1\sqrt{t^2 + 1} = (2t^2 + 1) + 2t\sqrt{t^2 + 1}$, so $x_2 = 2t^2 + 1$, $y_2 = 2t$ which satisfy $x_2^2 - (t^2 + 1)y_2^2 = 1$. You can show easily by induction that $x_n^2 - (t^2 + 1)y_n^2 = (-1)^n$. In fact, this prescription gives all the solutions in positive integers and the same idea solves the general Pell equation $x^2 - Ny^2 = \pm 1$. To make the calculations easier, we can use a recurrence for x_n and y_n. Note that

$$x_{n+1} + y_{n+1}\sqrt{t^2 + 1} = (x_1 + y_1\sqrt{t^2 + 1})(x_n + y_n\sqrt{t^2 + 1}) =$$
$$= x_1x_n + y_1y_n(t^2 + 1) + (x_1y_n + x_ny_1)\sqrt{t^2 + 1}.$$

Hence

$$x_{n+1} = tx_n + (t^2 + 1)y_n, \quad y_{n+1} = x_n + ty_n.$$

For later reference, note that $x_{n+1} - ty_{n+1} = y_n$. To relate this to the triangle equation $(p + tq)^2 - (t^2 + 1)q^2 = \pm 1$, we take $x = p + tq$, $y = q$, that is $q = y$, $p = x - ty$. Thus we can find infinitely many solutions p_n, q_n from the recurrence

$$p_1 = 0, \; q_1 = 1; \quad p_{n+1} = q_n, \; q_{n+1} = 2tq_n + p_n \quad (n = 1, 2, \ldots).$$

In particular, if $t = 1$, the recurrence becomes $p_{n+1} = q_n$, $q_{n+1} = 2q_n + p_n$ which is exactly the same as the recurrence obtained from the continued fraction for $\sqrt{2} - 1$. What happens when you start with $\sqrt{5}$, or $\sqrt{10}$, as suggested in Mr. Cohen's letter?

We managed to avoid continued fractions in this analysis because we were able to find a small solution of the equation $x^2 - (t^2 + 1)y^2 = -1$ by inspection. However, this is not always so easy. For example, the smallest solution of $x^2 - 61y^2 = 1$ is $x = 1766319049$, $y = 226153980$. Obstinate cases like this clearly need a better strategy than trial and error. We have already remarked that if $x^2 - Ny^2 = \pm 1$, then $\frac{x}{y}$ is a good rational approximation to $\sqrt{N}$. We can use this fact in reverse. First, find the continued fraction for $\sqrt{N}$ and then truncate it to get successive rational approximations $\frac{p}{q}$ for $\sqrt{N}$. We hope to find one of these approximations which satisfies $p^2 - Nq^2 = \pm 1$. In fact, all the solutions of Pell's equation can be found in this way. By way of illustration, let us find some integer solutions of the equation $x^2 - 7y^2 = 1$. We need the continued fraction expansion for $\sqrt{7}$. First write

$$\sqrt{7} = 2 + (\sqrt{7} - 2) = 2 + \frac{1}{\xi_1}$$

Here 2 is the largest integer less than $\sqrt{7}$, so the remainder $\sqrt{7} - 2$ is less than 1 and its reciprocal ξ_1 is greater than 1. We repeat this step and find the integer part of ξ_1.

$$\xi_1 = \frac{1}{\sqrt{7} - 2} = \frac{\sqrt{7} + 2}{(\sqrt{7})^2 - 2^2} = \frac{\sqrt{7} + 2}{3} = 1 + \frac{\sqrt{7} - 1}{3} = 1 + \frac{1}{\xi_2} .$$

Note how rationalising the denominator keeps the working simple. Again,

$$\xi_2 = \frac{3}{\sqrt{7} - 1} = \frac{3(\sqrt{7} + 1)}{7 - 1} = \frac{\sqrt{7} + 1}{2} = 1 + \frac{\sqrt{7} - 1}{2} = 1 + \frac{1}{\xi_3} ,$$

$$\xi_3 = \frac{2}{\sqrt{7} - 1} = \frac{2(\sqrt{7} + 1)}{7 - 1} = \frac{\sqrt{7} + 1}{3} = 1 + \frac{\sqrt{7} - 2}{3} = 1 + \frac{1}{\xi_4} ,$$

$$\xi_4 = \frac{3}{\sqrt{7} - 2} = \frac{3(\sqrt{7} + 2)}{7 - 4} = \sqrt{7} + 2 = 4 + (\sqrt{7} - 2) = 4 + \frac{1}{\xi_5} ,$$

and so on. But, if we are observant, we observe that $\xi_5 = \xi_1$, so the continued fraction is periodic. Thus

$$\sqrt{7} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ddots}}}}}}}}$$

The successive approximations to $\sqrt{7}$ are

$$\frac{p_0}{q_0} = 2, \quad \frac{p_1}{q_1} = 2 + \frac{1}{1} = 3, \quad \frac{p_2}{p_2} = 2 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{5}{2}, \; \ldots \; .$$

However, such calculations become a little tedious after a while, so let us think a little harder. The n-th approximation looks like

$$\frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_n}}}}$$

After some time, we hit upon the recurrence

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2} \qquad (n = 2, 3, \ldots)$$

It is easy to check this for $n = 2$, because

$$\frac{p_0}{q_0} = \frac{a_0}{1}, \quad \frac{p_1}{q_1} = \frac{a_0 a_1 + 1}{a_1} \quad \text{and} \quad \frac{p_2}{q_2} = \frac{a_0 a_1 a_2 + a_0 + a_2}{a_1 a_2 + 1} .$$

To prove our recurrence in general, we shall use induction. So suppose everything works for $n = 2, 3, 4, \ldots, k$. To find $\frac{p_{k+1}}{q_{k+1}}$, we write it, in a slightly tricky way, as

$$\frac{p_{k+1}}{q_{k+1}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \; a_{k-1} + \cfrac{1}{(a_k + \frac{1}{a_{k+1}})}}}}$$

that is, $\frac{p_{k+1}}{q_{k+1}}$ is the k-th approximation to the continued fraction with entries $a_0, a_1, \ldots, a_{k-1}, \; a_k + \frac{1}{a_{k+1}}$. The first $k - 1$ approximations of this funny continued fraction agree with those of our old continued fraction since the entries are the same up to a_{k-1}. But we can use our induction hypothesis to find the k-th approximation to the new continued fraction, so we find

$$\frac{p_{k+1}}{q_{k-1}} = \frac{(a_k + \frac{1}{a_{k+1}})p_{k-1} + p_{k-2}}{(a_k + \frac{1}{a_{k+1}})q_{k-1} + q_{k-2}} = \frac{(a_k p_{k-1} + p_{k-2}) + \frac{1}{a_{k+1}} p_{k-1}}{(a_k q_{k-1} + q_{k-2}) + \frac{1}{a_{k+1}} q_{k-1}} = \frac{p_k + \frac{1}{a_{k+1}} p_{k-1}}{q_k + \frac{1}{a_{k+1}} q_{k-1}} .$$

For the last step, we have used the induction hypothesis again. Thus

$$p_{k+1} = a_{k+1}p_k + p_{k-1}, \quad q_{k+1} = a_{k+1}q_k + q_{k-1}$$

completing our proof by induction. The calculations for $\sqrt{7}$ are conveniently done in a table

n	0	1	2	3	4	5	6	7	8	9	10	11
a_n	2	1	1	1	4	1	1	1	4	1	1	1
p_n	2	3	5	8	37	45	82	127	590	717	1307	2024
q_n	1	1	2	3	14	17	31	48	223	281	504	785
$p_n^2 - 7q_n^2$	-3	2	-3	1	-3	2	-3	1	-3	2	-3	1

Observe that the last row is also periodic and every fourth entry yields a solution to the equation $x^2 - 7y^2 = 1$. The general solution in positive integers could also be obtained from the first solution $x_1 = 8$, $y_1 = 3$, by

$$x_n + y_n\sqrt{7} = (8 + 3\sqrt{7})^n, \quad n = 1, 2, 3, \dots .$$

Try applying all this continued fraction machinery to $\sqrt{2}$, $\sqrt{5}$, $\sqrt{10}$, and then to $\sqrt{t^2 + 1}$.

What do the continued fractions have in common?

Are there any other irrational square roots with the same property?

What is the continued fraction expansion for $\sqrt{3}$?

Our machine produces "regular simple continued fractions", that is continued fractions in which the entries are all positive integers. Can you relate the regular simple continued fraction for $\sqrt{3}$ to the continued fraction in Mr. Cohen's letter?

What about $\sqrt{8}$, $\sqrt{15}$, ... ?

What is the pattern?

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