H.S.C. CORNER BY TREVOR

POLYNOMIALS

In the 1980 H.S.C. paper Question 9 in the 3 unit paper and Question 2 part (ii) in the 4 unit paper concerned polynomials which seemed to prove somewhat troublesome for students.

1. In the 3 unit paper we were given the polynomial

$$P(x) = p_0x^n + p_1x^{n-1} + \ldots + p_n$$

and was asked to determine the constants p_k in four different cases. Since each question should take less than 20 minutes, each case should be less than 5 minutes work, so it is worthwhile to think for a couple of minutes rather than attack each case routinely, slogging out masses of simultaneous equations. Here we give the solutions:

i) $P(x)$ is quadratic, $P(0) = 15$ and the minimum value of $P(x)$ is 3 when $x = 2$.

Since the minimum value of $P(x)$ is 3, when $x = 2$, we can write

$$P(x) = a(x - 2)^2 + 3$$

where a is yet to be found.

Since $P(0) = 15$, we have

$$P(0) = a(0 - 2)^2 + 3 = 4a + 3 = 15$$

and so $a = 3$ and

$$P(x) = 3(x - 2)^2 + 3$$
$$= 3x^2 - 12x + 15$$

ii) $P(x)$ is quadratic, $P(0) = 32$ and $P(2^t) = 0$ has roots $t = 1$ and $t = 3$. When $t = 1$, $2^t = 2$, when $t = 3$, $2^t = 8$. Therefore 2,8 are roots of $P(x)$. Hence

$P(x) = a(x - 2)(x - 8)$ and since $P(0) = 32$

$P(0) = a(0 - 2)(0 - 8) = 16a = 32$ and so

$a = 2$

$P(x) = 2(x - 2)(x - 8) = 2x^2 - 20x + 32$

iii) $P(x)$ has degree 4, has factors $(x+2)^2$ and $(x-2)^2$, and has remainder 50 on division by $x - 3$.

The given factors are quadratic whose product is a quartic, so

$$P(x) = a(x-2)^2(x+2)^2$$
$$= a(x^2-4)^2$$

But $P(3) = 50$, so $P(3) = a(9-4)^2 = 50$ and $a = 2$.

$P(x) = 2(x^2-4)^2 = 2x^4 - 16x^2 + 32$.

iv) $P(x)$ has degree 3, zeros at $x = 0$, $x = 1+\sqrt{2}$ and $x = 1-\sqrt{2}$, and $P(1) = 3$.

Obviously
$$P(x) = ax(x-(1+\sqrt{2})(x-(1-\sqrt{2}))$$
$$= ax(x-1-\sqrt{2})(x-1+\sqrt{2})$$
$$= ax((x-1)^2-2)$$
$$= ax(x^2-2x-1)$$

$P(1) = a.1.(1-2-1) = a.-2 = 3$, so $a = -\frac{3}{2}$ and

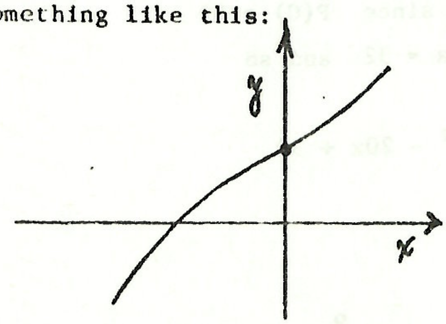
$$P(x) = -\frac{3}{2}(x^3-2x^2-x)$$
$$= -\frac{3}{2}x^3 + 3x^2 + \frac{3}{2}x$$

2. The 4 unit question was much harder. Here the polynomial given was $P(x) = x^5 - 5cx + 1$, where c is a real number and we had to prove that

a) If $c < 0$, $P(x)$ has just one real root, which is negative and

b) $P(x)$ has three distinct real roots if and only if $c > (\frac{1}{4})^{\frac{4}{5}}$.

a) This part of the proof could be regarded as just an application of Rolle's Theorem, i.e. for a polynomial (or any smooth, continuous function), there must be a maximum or a minimum between any two roots. Here $P'(x) = 5(x^4 - c)$. If $c < 0$ $P'(x) > 0$ for all x, and $P'(x)$ cannot be zero, hence $P(x)$ cannot have two zeros (roots). But the degree of $P(x)$ is odd (i.e. 5) therefore it must have one real root. Also $P(0) = 1 > 0$, and as $x \to \infty$ $P(x) \to \infty$ and as $x \to -\infty$ $P(x) \to -\infty$, hence the root must be negative. So for $c < 0$ $P(x)$ must look something like this:

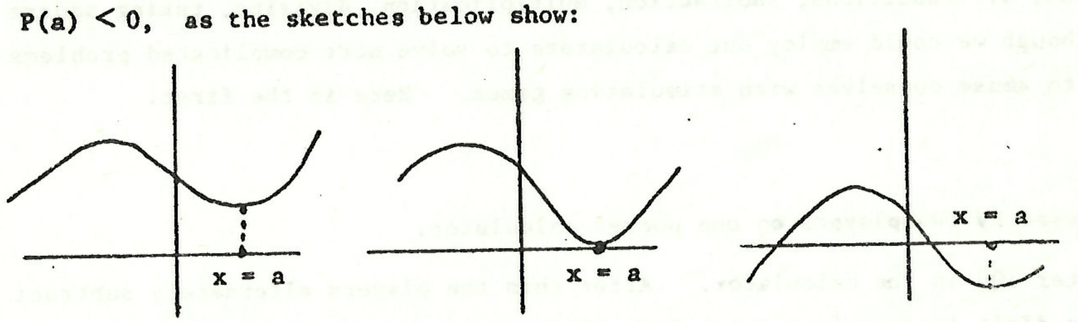


b) If $c > 0$ then $P'(x) = 5(x^4 - c) = 0$ when $x^2 = \pm c$. But $x^2 \neq -\sqrt{c}$ (as it has only complex solution), so $x^2 = \sqrt{c}$ and $x = \pm a$ where $a = c^{\frac{1}{4}}$.

Tabulating our results:

x	a	-a
$P(x) = x^5 - 5cx + 1$	$1 - 4a^5$	$1 + 4a^5$
$P'(x) = 5(x^4 - c)$	0	0
$P''(x) = 20x^3$	$P''(a) > 0$	$P''(-a) > 0$
	Minimum	Maximum

Thus $P(x)$ has a positive maximum at $x = -a$ and a minimum at $x = a$. There are three possibilities: either $P(a) > 0$, or $P(a) = 0$, or $P(a) < 0$, as the sketches below show:



Clearly the necessary and sufficient condition for three distinct roots is that

$$P(a) < 0,$$

$$1 - 4a^5 < 0$$

$$a^5 > \frac{1}{4}$$

i.e.

$$c^{\frac{5}{4}} > \frac{1}{4}$$

$$c > (\tfrac{1}{4})^{\frac{4}{5}}$$

That is: there are three distince roots if and only if, $c > (\frac{1}{4})^{\frac{4}{5}}$.

You might like to try the following (rather difficult) problems:

Let

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}$$

Prove that:

i) $P_2(x)$ has no real roots

ii) $P_3(x)$ has one real root

iii) $P_4(x)$ has no real roots

iv) In general $P_n(x)$ has no real zero if n is an even integer.

A solution will be in the next issue of Parabola.

COMPUTER - COLUMN

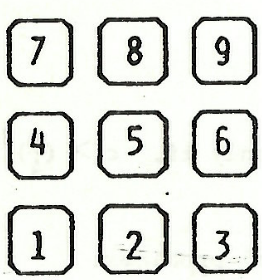
We are living at a time when more and more schools have a computer, and almost every student has a simple pocket-calculator. In our experience the majority of the students use their calculators only to carry out elementary operations, i.e. additions, subtraction, multiplication, division, taking square roots, though we could employ our calculators to solve more complicated problems and/or to amuse ourselves with stimulating games. Here is the first:

"LADDER".

Played by two players on one pocket calculator.

Enter 100 in the calculator. After this the players alternately subtract a single digit integer (not zero) from 100 according to the following rule: The first player may choose any number. After that the players must choose a button which is adjecent to the previous one on the calculator. For example: after 3 a player may push the 2, 5 or the 6 button. If somebody selects button 5, then clearly his opponent may choose any of the buttons except 5.

The loser is the one who gets the first negative result



This diagram shows the usual position of the digits on a calculator.

If you have access to a computer you could try to write a programme for this game. Use a smaller starting number, say 25, and try to find a winning strategy.