# THE KNIGHT'S TOUR

KIERAN LIM

Since the knight is the only piece that moves asymmetrically in chess, more problems have been based on the knight than on any other chess piece. The oldest such problem is that of the knight's tour and has been included in Arabic and Indian texts as early as the seventeenth century.

The problem is to find a path consisting of knight's moves that allows the knight to enter each square once. The number of solutions is huge and have not

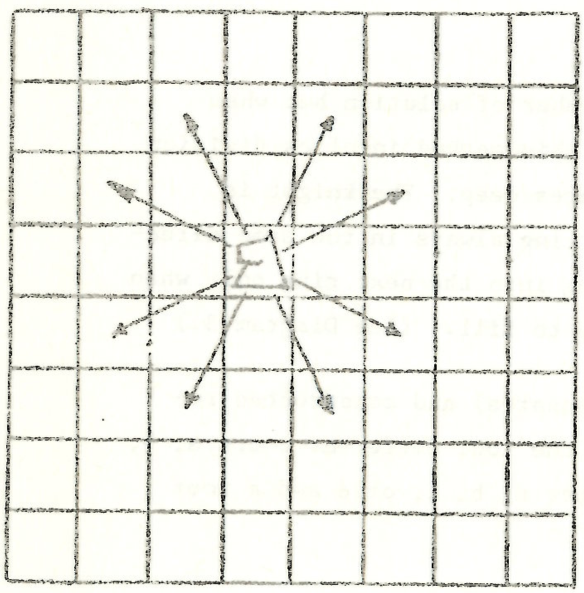
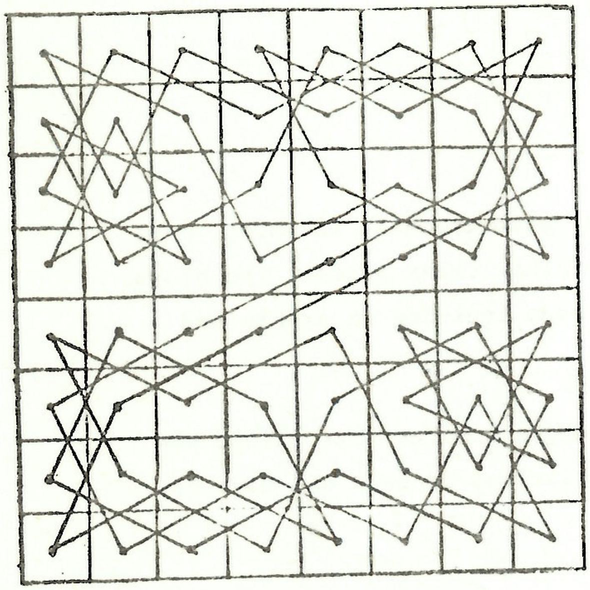


*Diagram 1:*
*Possible moves made by a knight in the centre of a chess board.*

been completely analysed. There are 122,802,512 "half-board" solutions alone! In this particular type of solution, the knight first covers half the board, a 4 × 8 rectangle, before entering the other half-board. Of the 122,802,512 half-board solutions, 7,763,536 are re-entrant, that is, the $64^{th}$ move would bring the knight back to the original square. 3,944 tours are symmetric and 4,064 have identical halves. Euler's solution shown in diagram 2 has identical halves and is re-entrant.

Werndorf's Rule is perhaps the easiest method of constructing a knight's tour. At every move, the knight is placed in the square from which there are the fewest exits to "unoccupied" squares. In most cases, a single false step, except in the last few moves, will not effect the result. However, the total number of solutions obtained by this method is not very great and are unsymmetrical and not re-entrant.



*Digram 2:*

*Euler's half-board solution.*

De Moivre's solution yields an even smaller number of solution but when drawn, de Moivre's solution is visually pleasing. This method involves division of the board into concentric "rings", each two squares deep. The knight is placed in the outermost ring and moves around that ring always in the same direction and filling all the squares of that ring, going into the next ring only when absolutely necessary. The innermost cells are easy to fill. (See Diagram 3.)

Roget divided the board into quarters (4 × 4 squares) and constructed re-entrant tours - in each quarter board. If we name the four different tours a, b, c, or d, the 64 of the board can be classified either a, b, c, or d and a tour

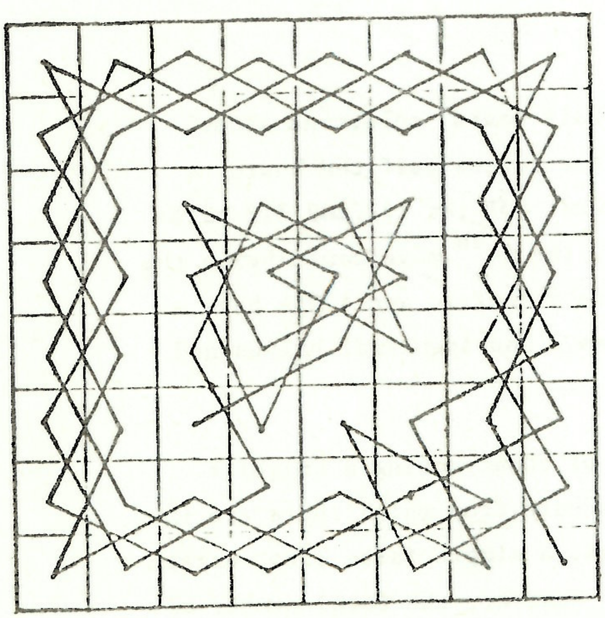


*Diagram 3:*

*A "de Moivre's solution".*

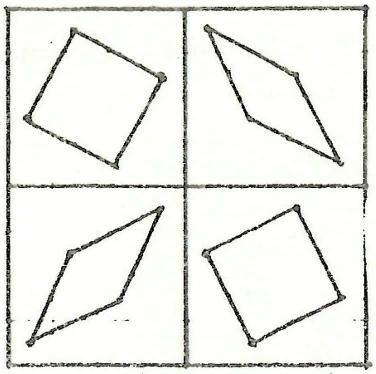


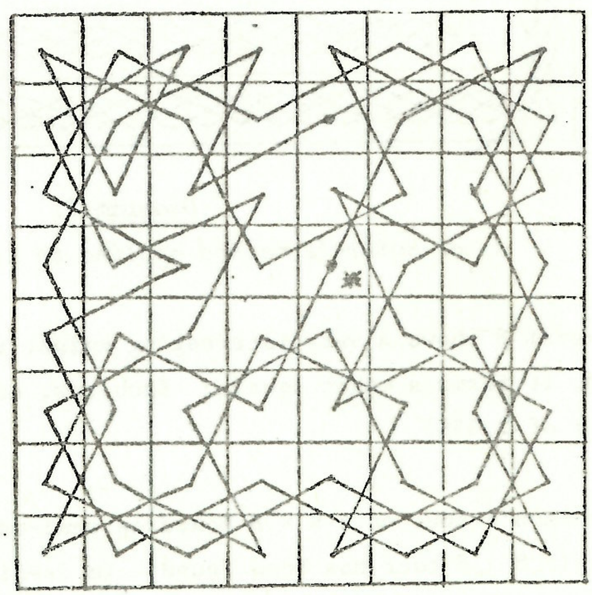
*Diagram 4:*

*Roget's four quarter board tours.*

constructed. The completed tour is shown in diagram 6. Starting in the cell marked *, the knight first covers all the "b" cells, then the "d" cells, followed by the "a" and the "c" cells.

Since the number of knight's tours on the 8 × 8 board is limited - the number is less then $^{168}C_{63}$ (approximately $10^{47}$), we can experiment with boards of other dimensions.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| a | c | d | b | a | c | d | b |
| d | b | a | c | d | b | a | c |
| c | a | b | d | c | a | b | d |
| b | d | c | a | b | d | c | a |
| a | c | d | b | a | c | d | b |
| d | b | a | c | d | b | a | c |
| c | a | b | d | c | a | b | d |
| b | d | c | a | b | d | c | a |



*Diagrams 5 and 6:*

*Construction of Roget's tour in quarter boards.*

In diagram 7, de Moivre's method is applied to a 25 × 25 board. The concentric rings are much more noticeable than on the 8 × 8 board.

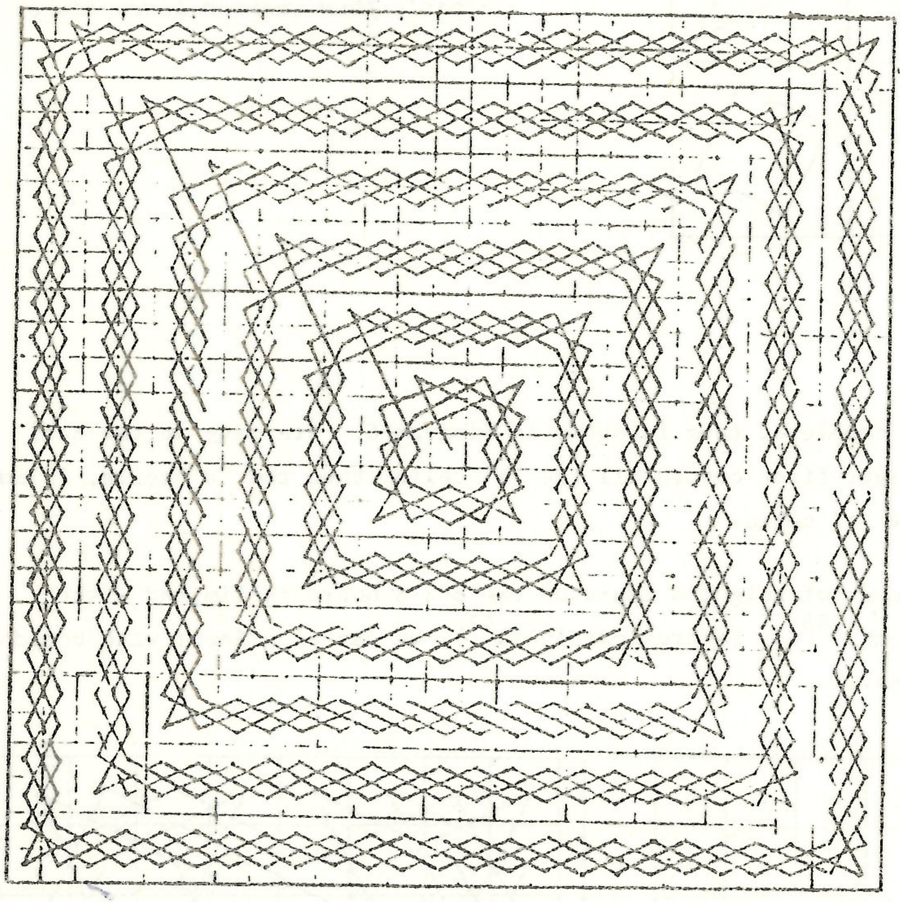


*Diagram 7:*
*de Moivre's method applied to a 25 × 25 board.*

Diagram 8 shows a very interesting knight's tour. When the moves have been *numbered*, it forms a magic square. Each row, column and main diagonal has a constant sum of 2,056!

Jaenisch's tour on a 8 × 8 board is semi-magic (see diagram 9). As yet, no fully magic 8 × 8 tour has been found. In Jaenisch's solutions only the rows and columns (not the diagonals) have the required constant of 260.

When experimenting with knight's tours on small rectangular boards, it should be noted that

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 182 | 217 | 170 | 75 | 188 | 219 | 172 | 77 | 228 | 37 | 86 | 21 | 230 | 39 | 88 | 25 |
| 169 | 74 | 185 | 218 | 171 | 76 | 189 | 220 | 85 | 20 | 229 | 38 | 87 | 24 | 231 | 40 |
| 218 | 183 | 68 | 167 | 222 | 187 | 78 | 173 | 36 | 227 | 22 | 83 | 42 | 237 | 26 | 89 |
| 73 | 168 | 215 | 186 | 67 | 174 | 221 | 190 | 19 | 84 | 35 | 238 | 23 | 90 | 41 | 232 |
| 182 | 213 | 166 | 69 | 178 | 223 | 176 | 79 | 226 | 33 | 82 | 31 | 236 | 43 | 92 | 27 |
| 165 | 72 | 179 | 214 | 175 | 66 | 191 | 224 | 81 | 18 | 239 | 34 | 91 | 30 | 233 | 44 |
| 212 | 181 | 70 | 163 | 210 | 177 | 80 | 161 | 48 | 225 | 32 | 95 | 46 | 235 | 28 | 93 |
| 71 | 164 | 211 | 180 | 65 | 162 | 209 | 192 | 17 | 96 | 47 | 240 | 29 | 94 | 45 | 234 |
| 202 | 13 | 126 | 61 | 208 | 15 | 128 | 49 | 160 | 214 | 130 | 97 | 148 | 243 | 132 | 103 |
| 125 | 60 | 203 | 14 | 127 | 64 | 193 | 16 | 129 | 112 | 145 | 242 | 131 | 102 | 149 | 244 |
| 12 | 201 | 62 | 123 | 2 | 207 | 50 | 113 | 256 | 159 | 98 | 143 | 246 | 147 | 104 | 133 |
| 59 | 124 | 11 | 204 | 63 | 114 | 1 | 194 | 111 | 144 | 255 | 146 | 101 | 134 | 245 | 150 |
| 200 | 9 | 122 | 55 | 206 | 3 | 116 | 51 | 158 | 253 | 142 | 99 | 154 | 247 | 136 | 105 |
| 121 | 58 | 205 | 10 | 115 | 54 | 195 | 4 | 141 | 110 | 155 | 254 | 135 | 100 | 151 | 248 |
| 8 | 199 | 56 | 119 | 6 | 197 | 52 | 117 | 252 | 157 | 108 | 139 | 250 | 153 | 106 | 137 |
| 57 | 120 | 7 | 198 | 53 | 118 | 5 | 196 | 109 | 140 | 251 | 156 | 107 | 138 | 249 | 152 |

*Diagram 8:*

*A fully magic re-entrant 16 × 16 knight's tour.*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 63 | 22 | 15 | 40 | 1 | 42 | 59 | 18 |
| 14 | 39 | 64 | 21 | 60 | 17 | 2 | 43 |
| 37 | 62 | 23 | 16 | 41 | 4 | 19 | 58 |
| 24 | 13 | 38 | 61 | 20 | 57 | 44 | 3 |
| 11 | 36 | 25 | 52 | 29 | 46 | 5 | 56 |
| 26 | 51 | 12 | 33 | 8 | 55 | 30 | 45 |
| 35 | 10 | 49 | 28 | 53 | 32 | 47 | 6 |
| 50 | 27 | 34 | 9 | 48 | 7 | 54 | 31 |

*Diagram 9:*

*Jaenisch's semi-magic re-entrant solution.*

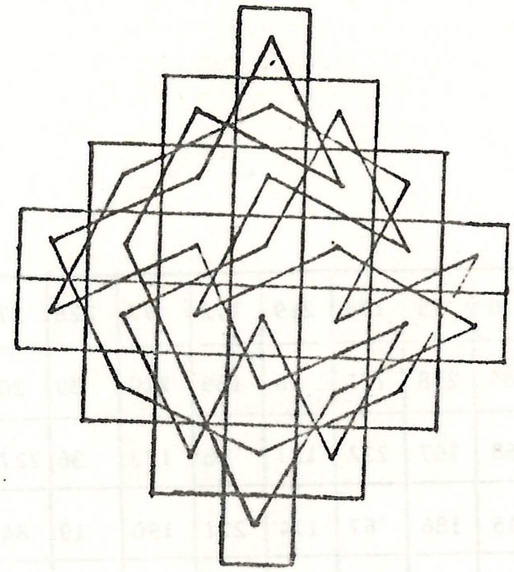


*Diagram 10:*

*Tour on an irregular board.*

(a) if one dimension is less than 3, no tour is possible;

(b) if one dimension is 3, the other must be ≥ 7. If the second dimension is even and ≥ 10, re-entrant solutions exist;

(c) if one dimension is 4, no re-entrant solutions exist;

(d) tours are impossible on a 4 × 4 board;

(e) if one dimension is 5, the other must be ≥ 5. If the second dimension is even, re-entrant solutions exist;

(f) if one dimension is 6, the other must be ≥ 5. Solutions will be re-entrant;

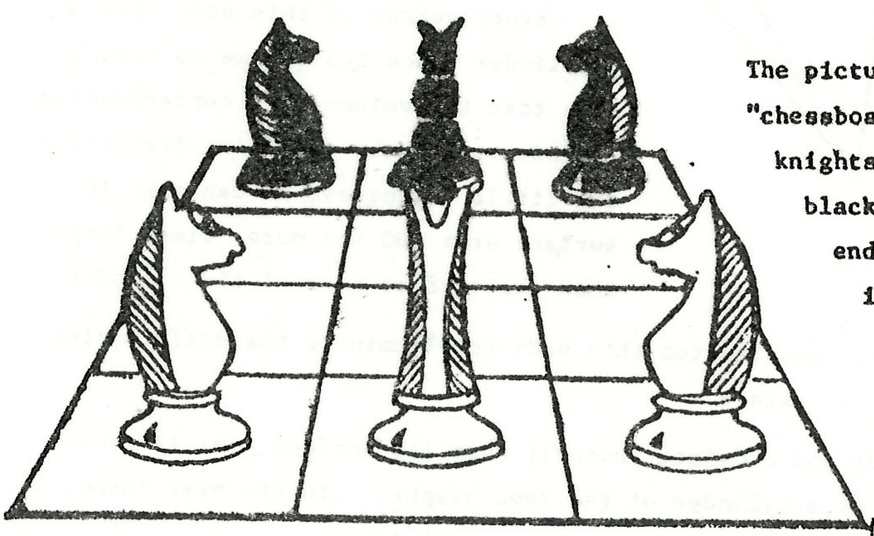
(g) if one dimension is ≥ 7, a tour can always be found.

Perhaps you may like to try your hand at constructing some knight's tours. Why not try some irregular shapes (see diagram 10)? Or try to construct half-board solutions or impose other restrictions such as making a tour re-entrant or semi-magic? Or even a magic tour!

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 46 | 55 | 44 | 19 | 58 | 9 | 22 | 7 |
| 43 | 18 | 47 | 56 | 21 | 6 | 59 | 10 |
| 54 | 45 | 20 | 41 | 12 | 57 | 8 | 23 |
| 17 | 42 | 53 | 48 | 5 | 24 | 11 | 60 |
| 52 | 3 | 32 | 13 | 40 | 61 | 34 | 25 |
| 31 | 16 | 49 | 4 | 33 | 28 | 37 | 62 |
| 2 | 51 | 14 | 29 | 64 | 39 | 26 | 35 |
| 15 | 30 | 1 | 50 | 27 | 36 | 63 | 38 |

*Diagram 11:*
*Another semi-magic tour.*

**EDITOR'S COMMENT:** Kieran Lim is a first year University student now, he was one of our most prolific problem solvers and contributors. We would like to thank him for his continued interest.



The picture shows a 3 × 4 "chessboard" with three white knights at one end and three black knights at the other end. The problem is to interchange the white knights with the black knights. The knights move in the usual manner. It is not necessary to move white and black alternatively. Can you find a solution using no more than 17 moves?

(From the "Mathematical Digest".)