SOLUTIONS TO PROBLEMS FROM VOLUME 16, NUMBER 2.

Q. 455. The rule for leap years runs as follows: A year which is divisible by 4 is a leap year except that years which are divisible by 100 are not leap years unless they are divisible by 400. Now, January the first, 1980, was a Tuesday. Prove that January the first in a new century (that is, 1900, 2000, 2100, ...) is never a Tuesday. What days of the week can it be?

SOLUTION: Since $7 . 52 = 364$, an ordinary year exceeds a whole number of weeks by 1 day, a leap year exceeds a whole number of weeks by 2 days. Hence the period from 1/1/1980 to 1/1/2000 is a whole number of weeks plus $(5.2 + 15.1)$ days, which is 4 days more than a whole number of weeks. Hence 1/1/2000 will be a Saturday. Similarly, since $(25.2 + 75.1) = 7 \times 17 + 6$, the period from 1/1/2000 to 1/1/2100 is 6 days more than a whole number of weeks, so that 1/1/2100 is a Friday; and centuries beginning with a year not a multiple of 400 exceed a whole number of weeks by 5 days (since there is one fewer leap year) whence 1/1/2200 falls on a Wednesday, 1/1/2300 on a Monday, and 1/1/2400 again on a Saturday. Since this is the same day of the week as 1/1/2000 the pattern Saturday, Friday, Wednesday, Monday repeats indefinitely for the first days of successive centuries.

Correct Solutions from: K. Lim (St. Ignatius College); D. Everett (Kotara High School); S.S. Wadhwa (Ashfield Boys' High School); N. Brown (Dickson College).

Q. 456. Find the last two digits of 9^{10} in decimal notation. Find the last three digist of $9^{9^{9}}$.

SOLUTION: A few minutes calculation obtains the last three digits only of 9^n for $n = 1, 2, 3, \ldots, 10$ viz

$9^1 = 9$; $9^2 = 81$; $9^3 = 729$; $9^4 = \ldots 561$; $9^5 = \ldots 049$; $9^6 = \ldots 441$;
$9^7 = \ldots 969$; $9^8 = \ldots 721$; $9^9 = \ldots 489$; $9^{10} = \ldots 401$.

In particular the last two digits of 9^{10} are 01 .... (1)

Now the last three digits of $9^{20} = (\ldots 401) \times (\ldots 401)$ are easily calculated to be ...801; similarly the last three digits of 9^{30} are ...201; of 9^{40} are ...601; and of 9^{50} are ...001.

It follows that the last 3 digits of 9^N are the same as the last three digits of 9^R, where $N = 50Q + R$. Take $N = 9^{9^9} = 9^K$. Since $K = 9^9$ ends in a 9, $K = 10.U + 9$ and the last two digits of $9^K = (9^{10})^U.9^9$ are ...89, the same as the last two digits of 9^9, because of (1). Hence $N = 50Q + 39$, so that the last 3 digits of 9^N are the same as the last 3 digits of $9^{39} = 9^{30}.9^9 = (\ldots 201) \times (\ldots 489) = \ldots 289$.

Correct Solutions from: D. Everett (Kotara High School); R. Youhana (North Sydney Boys' High School); S.S. Wadhwa (Ashfield Boy's High School); K. Lim (St. Ignatius' College).

Several point out that the last three $9^{9^{9^{9^{9^{9}}}}}$ are ...289 provided there are at least 3 nines in the tower.

Q. 457. Find all pairs of positive numbers x,y such that $(x^2 + 4y^2)/xy = 5$ and $x^y = 4$.

SOLUTION: $\dfrac{x^2 + 4y^2}{xy} = 5 \Rightarrow x^2 - 5xy + 4y^2 = 0 \Rightarrow x = y$ or $x = 4y$.

Now $x^y = 4$ and $x = y \Rightarrow x^x = 4 = 2^2$ which has the obvious solution $x = y = 2$. Again $x^y = 4$ and $y = \frac{x}{4} \Rightarrow x^{x/4} = 4 \Rightarrow x^x = 4^4$, with the obvious solutions $x = 4$, $y = 1$.

To show that there are no other solutions it would suffice to observe that the function x^x has values less than 1 for $0 < x < 1$, and is an obviously increasing function for $x \geq 1$, whence $x = 2$ is the only solution of $x^x = 4$, and $x = 4$ is the only solution of $x^x = 4^4$. An excellent alternative argument was supplied by R. Youhana, who showed that the graph $y = \dfrac{\ln 4}{\ln x}$ $(x > 0)$ intersects each of the straight lines $y = x$ and $y = 4x$ at one point only.

Correct Solutions from: R. Youhana (North Sydney Boys' High School); K. Lim (St. Ignatius' College); D. Everett (Kotara High School); S.S. Wadhwa (Ashfield Boys' High School).

Q. 458. A recent archeological find contains a partly obliterated description of a rather astonishing ancient Persian relay race between two teams of two members each in which a handicap was arrived at by the following process. Each of the four runners took a handful of wheat. After pairing off into teams, each team counted its combined wheat grains into x cups (the number x is no longer decipherable), each cup containing the same number of grains, until there were too few grains remaining to again place one in each cup. If a team was left with r grains, it did not start running until the count of r after the starting signal. In one such event, the four athletes Artaxerxes, Belshazzar, Cyrus and Darius (henceforth referred to by their initials) took handfuls of wheat containing 8965, 7672 and 8512 grains respectively. It is reported that C said to A: "I suggest we pair together as a team, since then, you see, there would be no grains of wheat left over. We would start right on the singal." To this, A replied: "Unfortunately, a minute ago, B addressed me in identical terms and I accepted his offer." In the ensuing race, in spite of being considerably handicapped by having to start on the count of y (also obliterated), C and D emerged victorious. Can you decide what were the obliterated numbers x and y?

<u>SOLUTION</u>: From the information given x is a common factor of $(8965 + 7672) = 16637$ and $8965 + 8434 = 17399$; hence also of the difference $762 = 2 \times 3 \times 127$. Either by completing the Euclidean Algorithm (D. Everett, Kotara High School) or by observing that neither 2 nor 3 is a factor of 16637 or 17399, (R. Youhana, North Sydney Boys' High School) the h.c.f. of 16637 and 17399 is 127, a prime number. Therefore $x = 127$ and y is the remainder when $(7672 + 8512) = 16184$ is divided by 127; viz $y = 55$.

Also solved by S.S. Wadhwa (Ashfield Boys' High School) and N. Brown (Dickson College).

<u>Q. 459</u>. For an integer n written in decimal notation, we define the number $S(n)$ to be the sum of the units, hundreds, ten thousands, ... digits of n, plus ten times the sum of the tens, thousands, hundred thousands, ... digits. For example, $S(123456789) = (9 + 7 + 5 + 3 + 1) + 10(8 + 6 + 4 + 2) = 225$. Show that $n - S(n)$ is always a multiple of 99. Find $S(S(S(1980^{1980})))$.

<u>SOLUTION</u>: Let $n = a_0 + 10.a_1 + 10^2.a_2 + 10^3.a_3 + \dots + 10^k.a_k$ where we can assume that k is odd by allowing $a_k = 0$ if necessary.

Then $S(n) = (a_0 + 10.a_1) + (a_2 + 10.a_3) + (a_4 + 10.a_5) + \dots$

$\therefore \quad n - S(n) = (10^2 - 1)(a_2 + 10.a_3) + (10^4 - 1)(a_4 + 10.a_5) + \ldots$

$+ (10^{k-1} - 1)(a_{k-1} + 10.a_k).$

Since $10^{2t} - 1 = 99 \ldots 99 = 99 \times (10101 \ldots 01)$, every term on the R.H.S. is a multiple of 99, whence so is $n - S(n)$. Hence $S(n)$ and n leave the same remainder on division by 99. Since $x = 1980^{1980}$ is clearly a multiple of 99, so is $S(x)$, and then $S(S(x))$. Finally $S(S(S(x)))$ is some (obviously positive) multiple of 99.

We now make a rough upper estimate of the size of $S(S(S(x)))$. $x = 1980^{1980}$ $(< 10000^{2000})$ has fewer than 8000 digits so that $S(x) < (4000 \times 9) + 10(4000 \times 9) < 400000$.
Hence $S(S(x)) < 3 \times 9 + 10(9 + 9 + 3) < 300$ and $S(S(S(x)) < 9 + 10(9 + 2) < 120$.
Since the only positive multiple of 99 less than 120 is 99 it follows that $S(S(S(1980^{1980}))) = 99$.

Correct solutions from: D. Everett (Kotara High School); R. Youhana (North Sydney Boys' High School); N. Brown (Dickson College); S.S. Wadhwa (Ashfield Boys' High School). K. Lim (St. Ignatius College) and R. Bozier supplied correct solutions of the first part, but incomplete working for the second.

Q. 460. At a party, there are 2n people, of whom each one is acquainted with at least n others. Prove that it is possible to seat them at a circular table so that each is seated between two acquaintances.

SOLUTION: (1) We first prove that it is possible to seat at least n + 1 round the table, each flanked by 2 acquaintances. Stand any person on a line, next to him place an acquaintance, then one of his acquaintances, and so on, adding at either end of the line until it is impossible to obtain a longer line in which each person knows both his neighbours. Call the person at one end of the line A. Then all of A's acquaintances are already in the line. The section of the line from A to the most distant of his friends therefore contains at least n + 1 people, and these may be seated round the table in the order in which they stand in the line.

(2) Suppose now that k is the largest number of people which can be seated round the table so that each knows both neighbours. Because of (1)

$k \geqslant n + 1$. We are asked to prove that $k = 2n$. Suppose that $k < 2n$. If B is one of the remaining $2n - k$ people, since $2n - k \leqslant n - 1$ some of B's acquaintances must already be seated. We can stand B behind a seated acquaintance, P, then (if there is one standing) another acquaintance of B behind him and so on, to obtain a queue of people standing behind one of the seated group so that each person in the queue knows his neighbours and so that the queue cannot be lengthened further. Suppose there are ℓ people standing in the queue, where $\ell \leqslant 2n - k$. If the person at the tail of the queue is C, all of his acquaintances must already be either seated, or in front of him in the queue. A simple calculation shows that C must have at least $n - (\ell - 1) - 1 = (n - \ell)$* acquaintances other than P who are already seated. Let S be the set of ℓ people seated to the right of P. None of them (D say) could be an acquaintance of C since otherwise we could replace the people between P and D by the people standing in the queue to obtain more than k people seated satisfactorily, contradicting the maximality of k. The same argument of course applies to the ℓ people T seated to the left of P. Hence all of C's seated acquaintances are included amongst the k - 2
$k - 2\ell - 1$ people, U, more than ℓ seats away from P. Furthermore no two adjacent people seated can both be acquainted with C, since we could simply seat C between them to again contradict the maximality of k. If at most every second person in U is acquainted with C, his seated acquaintances in the arc are in number $\leqslant \dfrac{(k - 2\ell - 1) + 1}{2}$

$\therefore$ Using *, $n - \ell \leqslant \frac{k}{2} - \ell$

$\therefore$ $k \geqslant 2n$

Hence we must have $k = 2n$ as required.

This was much more difficult than I had expected. Perhaps there is a simpler argument, but no correct attempts were received.

Q. 461. Let $p(1), p(2), \ldots, p(n)$ be the first n primes. Partition these primes into two sets A and B, so that A consists of some of the primes and B consists of all the others. Let a be the product of the primes in A and let b be the product of the primes in B. It is claimed that $m = |a - b|$ is always either 1 of a prime. Is this true? If it is true, prove it. If not, find the smallest n for which the claim is false and the smallest composite number m obtainable in this way.

SOLUTION: K. Lim (St. Ignatius College) writes:-

"The claim that m is 1 or a prime is false. Consider the first 5 primes {2, 3, 5, 7, 11}:

$$2 \times 3 \times 5 \times 11 - 7 = 323 = 17 \times 19.$$

5 is in fact the least numbers of primes for which a composite difference can be obtained.

$$221 = (3 \times 7 \times 11) - (2 \times 5) = 13 \times 17$$

is the smallest such composite difference."

COMMENT: Since it is obviously merely a matter of trial to verify the above statements, the above must be accepted as a correct answer. However the amount of labour involved is reduced by the following observation, which is therefore worth adding:- A difference so obtained starting with n primes, cannot be a multiple of any of those primes; hence if composite it must exceed $(p_{n+1})^2$. If $n = 4$, $p_5 = 11$ and since the largest possible difference is $3.5.7 - 2 = 103 < 121$ no composite difference is possible.

Q. 462. You are given five weights of identical appearance weighing respectively 1, 2, 3, 4, and 5 grams. Show how, with an accurate balance but no other standard weights, it is possible to sort out which is which in only five weighings.

SOLUTION: Label the weights A, B, C, D, and E. Take for the first three weighings:

1. A and B v C and D
2. A and C v B and D
3. A and D v B and C.

If no balance was obtained, E must have been either 2 grams or 4 grams; if E was 1, 3 or 5 grams a balance is obtained in one of the three weighings.

Case 1. weight of E odd.

We can suppose without loss of generality that the results of the weighing were 1 balance (Bal), 2 and 3 left heavy (L). [For example, if the results were in fact 1. L, 2. R, 3. Bal; i.e. 1. left heavy 2, right heavy etc; - then relabelling the weights by changing the label B into A, C into B, A into D, and D into C would reestablish the relationships between the weights A, B, C, and D which we are assuming.]

There are now 6 possibilities

1 (a)	E = 1 gm	:	A = 5,	D = 2,	B = 3,	C = 4
1 (b)	E = 1 gm	:	A = 5,	D = 2,	B = 4,	C = 3
1 (c)	E = 3 gm	:	A = 5,	D = 1,	B = 2,	C = 4
1 (d)	E = 3 gm	:	A = 5,	D = 1,	B = 4,	C = 2
1 (e)	E = 5 gm	:	A = 4,	D = 1,	B = 2,	C = 3
1 (f)	E = 5 gm	:	A = 4,	D = 1,	B = 3,	C = 2 .

One can now finish off as follows:

4th weighing B v C

5th weighing A v E and the lighter of B and C;

since no two of 1(a), ..., 1(f) yields the same outcome for both these weighings.

Case 2. E is 4 grams; none of the first three weighings produced a balance. One of the weights (viz the 5 gram weight) was always on the heavier side in the first three weighings; but there was no weight which was always on the lighter side. For definiteness, let us suppose that the result of the first three weighings was L, R, R. Then E = 4 grams, and B = 5 grams. It ramains to determine the weights of A, C and D. This can be accomplished as follows:-

4 th weighing E v A + C (D is 1, 2 or 3 grams according as the result of this weighing is L, Bal, or R).

5 th weighing A v C.

Case 3. None of the first three weighings balanced; one weight was always on the light side, but no weight always on the heavy side. For definiteness, suppose R, R, R.

Then E = 2 grams, and A = 1 gram. Proceed as follows

4 th weighing B v C

5 th weighing D and A v the heavier of B and C.

For definiteness suppose C is the heavier of B and C. The three possibilities

(a) B = 3, C = 4, D = 5

(b) B = 3, C = 5, D = 4

(c) B = 4, C = 5, D = 3

all yield different results for the fifth weighing.

Q. 463. In the preceding question, show that you cannot guarantee to identify the weights in only four weighings.

SOLUTION: We have to decide which of the $5! = 120$ possible arrangements of the 5 weights actually applies. But in 4 weighings, in each of which only 3 results are possible (balance, left heavy, or right heavy) there are only $3^4 = 81$ different outcomes. Hence more than one arrangement of the weights must yield the same outcome for at least one outcome.

Correct solutions from: K. Lim (St. Ignatius College); and N. Brown (Dickson College).

Q. 464. A number of circles are drawn in the plane. There are exactly 12 points on the plane which lie on more than one circle. What is the smallest possible number of regions into which the plane could be subdivided by these circles?

SOLUTION: The minimum number of regions is 14. It is possible to prove by induction that if there are points on more than one circle the minimum number of regions is $n + 2$. This is obvious if $n = 1$ since we must have at least two circles touching at the point, yielding at least 3 regions.

Suppose the statement above is known to be true for $n = 1, 2, 3, \ldots k$ but false for $n = k + 1$ i.e. there is a collection of circles, with $k + 1$ points lying on more than one circle, which divides the plane into fewer than $k + 3$ regions. Take one of the circles whose removal does not increase the number of disconnected groups of circles in the figure. (That such a circle exists is intuitively clear, and a formal proof, again by induction, is easy to find). If m of the $(k + 1)$ points lie on this circle, they divide the circumference into m arcs. As each arc, say PQ, is deleted the regions on either side of it coalesce. (They cannot be the same region already, since that would mean that the part of the figure containing P has now been separated from the point containing Q). Thus removal of the m arcs reduces the number of regions by m. The number of points still lying on more than one circle will have decreased by at most m.. (We can assume it has decreased by at least one since otherwise we can delete another circle in the sameway, reducing the number of regions still further). Hence we are left with a collection of circles dividing the plane into fewer than $k - m + 3$ regions and with at least $k + 1 - m$ points lying on more than one circle, which contradicts the induction hypothesis. This contradiction completes the proof.

Correct answers from: K Lim (St. Ignatius College); R. Youhana (North Sydney Boys' High School); D. Everett (Kotara High School); and N. Brown (Dickson College).

Q. 465. Let AB be a line segment whose midpoint M is marked, and let P be a point in the plane not on AB. Show how, using no drawing instrument except a straight edge (and a pencil), you can construct a line through P parallel to AB.

SOLUTION: A. Jenkins (North Sydney Boys' High School) writes: "First draw the line AP extending it past P to an arbitrary point Z. Join ZB, PB and ZM, calling the intersection of PB and ZM, X. Draw AX, extending it to intersect ZB at Y. Then PY is parallel to AB."

R. Bozier (Barker College)and N. Brown (Dickson College) also sent in this construction, the latter gave the following proof.

One method of proving this assertion is the following:-

$$\frac{ZP}{PA} = \frac{\text{Area of } \Delta PZB}{\text{Area of } \Delta PAB} \quad \text{(since the heights to the vertex B from the base line AZ are the same)}$$

$$= \frac{\text{Area of } \Delta PZX}{\text{Area of } \Delta PAX} \quad \text{(same argument, X replacing B)}$$

$$= \frac{\text{Area of } \Delta PZB - \text{Area of } \Delta PZX}{\text{Area of } \Delta PAB - \text{Area of } \Delta PAX} = \frac{\text{Area } \Delta ZXB}{\text{Area } \Delta AXB} .$$

Similarly $\frac{ZY}{YB} = \frac{\text{Area } \Delta ZXA}{\text{Area } \Delta AXB}$, and $\frac{AM}{MB} = \frac{\text{Area } \Delta AXZ}{\text{Area } \Delta BXZ}$.

Since AM = MB, Area ΔAXZ = Area ΔBXZ, whence $\frac{ZP}{PA} = \frac{ZY}{YB}$. Hence, since PY divides ZA and ZB in the same ratio, PY ∥ AB.

(For those acquainted with Ceva's Theorem, this proof can be greatly shortened)

Mathematicians have inherited from the Greek geometers strict rules for what is and is not allowed when using a drawing instrument. With a straightedge, the only operation permitted is to draw a straight line passing through up to two specified points. It is not permitted to transfer or to measure, lengths as with a graduated ruler. Other attempts sent to me transgressed in this respect.

Q. 466. Given a point P and three distances x, y and z, Show how to construct an equilateral triangle ABC, using only straightedge and compass, so that P is inside the triangle and PA = x, PB = y and PC = z.

SOLUTION: Analysis. Suppose ABC is equilateral equilateral with PA = x, PB = y and PC = z. Construct Q such that APQ is equilateral (see figure 1) and join CQ. Then ΔAPB is congruent to ΔAQC.

Since AP = AQ

AB = AC

and $\angle PAB$ (θ_1) = $\angle QAC$ (θ_2) (both $= 60^\circ - \alpha$)

$\therefore$ QC = PB = y.

Now the triangle PQC is seen to have sides of length x, y and z. Thus a necessary condition for the construction to be possible is that the sum of the shortest two lengths given exceeds the greatest length. Provided this is so we can construct the figure above, given P, by drawing any line segment PQ of length x. On one side of PQ construct the triangle PCQ using the given lengths y and z. On the other construct the equilateral triangle APQ. Now CA is one side of the desired equilateral triangle. Construct ABC equilateral with B on the same side of AC as P.

Correct solution from: N. Brown (Dickson College).