H.S.C. CORNER BY TREVOR

PROJECTILES

The differential and integral Calculus has important applications in physics and/or mechanics. One such problem involves projectiles - the concern of these notes - which you may encounter in the H.S.C. papers.

For the sake of simplicity we take the origin to be the point of projection, x,y the horizontal and vertical co-ordinates respectively, and u, v the velocities of the projectile in the x,y directions respectively at time t after the instant of projection. Let V be the speed of projection at $t = 0$, α the angle of projection at $t = 0$, then the familiar equations of motion, neglecting air resistance, are

$$x = Vt\cos\alpha \qquad (1)$$

$$y = Vt\sin\alpha - \tfrac{1}{2}gt^2 \qquad (2)$$

The path of the projectile is obtained by eliminating t from (1) and (2), yielding

$$y = x\tan\alpha - \tfrac{1}{2}gx^2\sec^2\alpha/V^2 \qquad (3)$$

This is the equation of the parabola which intersects the x axis at $x = 0$ and at $x = R$, where R, the "range", is

$$R = \frac{2V^2\sin\alpha\cos\alpha}{g} = \frac{V^2\sin 2\alpha}{g} \qquad (4)$$

It can also be shown from (1), (2) and (3) that the 'time of flight' is $2V\sin\alpha/g$, and that the highest point reached by the projectile has a height of $V^2\sin^2\alpha/2g$, and this point is reached after a time of $V\sin\alpha/g$.

Most problems on projectiles can be solved by the application of equations (1), (2), (3) and (4). Frequently V is taken as constant and α may be varied. Regarding R as a function of α we observe from (4) that the "maximum range" occurs when $\sin 2\alpha$ is maximum, i.e. when $\sin 2\alpha = 1$ and so $\alpha = \frac{\pi}{4}$. Hence

$$R_{max} = \frac{V^2}{g}.$$

If we wish to reach a point at a distance d on the ground, it is clear that the condition is

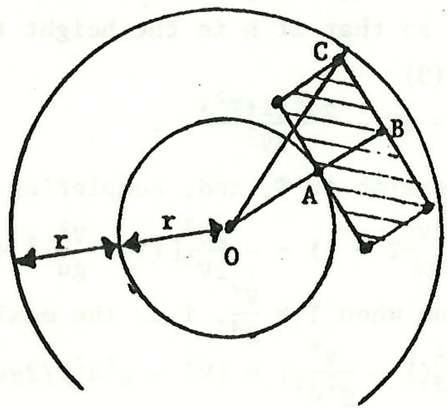
$$d \leq R_{max}, \text{ i.e. } \quad d \leq \frac{V^2}{g}.$$

From recent H.S.C. papers we discuss two questions.

(i) In 1979, 4 Unit Question 3 requires a proof of (4). It then goes on "*A garden sprinkler sprays water symmetrically about its vertical axis at a constant speed of* $V\ ms^{-1}$. *The initial direction of spray varies continuously between angles of* $15°$ *and* $60°$ *to the horizontal. Prove that from a fixed position O on level ground, the sprinkler will wet the surface of an annular region of centre O with internal and external radii* $\frac{V^2}{2g}$ *m and* $\frac{V^2}{g}$ *m respectively. Deduce that by locating the sprinkler appropriately relative to a garden bed of size 6m by 3m, the entire bed may be watered provided that* $\frac{V^2}{2g} \geqslant 1 + \sqrt{7}$."

The first part is easy. The maximum range is $\frac{V^2}{g}$, and therefore the external radius of the annulus is $\frac{V^2}{g}$. The internal radius is the lesser of $\frac{V^2 \sin 30°}{g}$ and $\frac{V^2 \sin 120°}{g}$, and this is $\frac{V^2}{2g}$.

The last part is an exercise in geometry, suitably disguised. Let $\frac{V^2}{2g} = r$, and place the rectangle in the annulus, as in the figure:



Clearly the best position is the symmetrical one. Then OB = OA + AB = r + 3, and $OC^2 = 3^2 + (r+3)^2 = r^2 + 6r + 18$. But $OC \leqslant 2r$,

$\therefore\ r^2 + 6r + 18 \leqslant 4r^2$, i.e. $3r^2 - 6r - 18 \geqslant 0$, so that

$$r^2 - 2r - 6 \geqslant 0.$$

Solving the inequality, either $r \leqslant 1 - \sqrt{7}$ or $r \geqslant 1 + \sqrt{7}$. But $r \geqslant 0$ and $1 - \sqrt{7} < 0$, and hence $r = \frac{V^2}{2g} \geqslant 1 + \sqrt{7}$.

(ii) In 1980, 4 Unit Question 4 asks for (3) and (4), then it goes on: "*A vertical wall is a distance d from the origin, and the plane of the wall is perpendicular to the plane of the particle's trajectory.*

(a) Show that if $d < \frac{V^2}{g}$, *the particle will strike the wall provided that*

$$\beta < \alpha < \frac{\pi}{2} - \beta,$$

where $\beta = \frac{1}{2} \sin^{-1}(\frac{gd}{V^2})$

(b) Show also that the maximum height the particle can reach on the wall is $(V^4 - g^2d^2)/2gV^2$*."*

(a) First of all, $d < \frac{V^2}{g}$, otherwise the wall is not within range at all, i.e. d < maximum range. When $d < \frac{V^2}{g}$, then, to strike the wall

$$R > d,$$

i.e.

$$\frac{V^2 \sin 2\alpha}{g} > d$$

$$\therefore \sin 2\alpha > \frac{gd}{V^2}.$$

This implies that

$$\sin^{-1}(\frac{gd}{V^2}) < 2\alpha < \pi - \sin^{-1}(\frac{gd}{V^2})$$

and the result follows.

(b) This part looks difficult but is really very simple if we put $T = \tan \alpha$, and then $\sec^2\alpha = 1 + T^2$, so that if h is the height the particle reaches on the wall, then, from (3)

$$h = dT - \frac{gd^2(1+T^2)}{2V^2}.$$

This is a quadratic expression in T, and, completing the square,

$$h = -\frac{gd^2}{2V^2}(T^2 - \frac{2V^2}{gd}T + 1) = -\frac{gd^2}{2V^2}[(T - \frac{V^2}{gd})^2 + 1 - \frac{V^4}{g^2d^2}]$$

This expression is maximum when $T = \frac{V^2}{gd}$, i.e. the maximum height reached is

$$-\frac{gd^2}{2V^2}(1 - \frac{V^4}{g^2d^2}) = (V^4 - g^2d^2)/2gV^2.$$

SOLUTION OF THE POLYNOMIAL PROBLEM IN VOLUME 17, 1.

Let $P_n(x) = 1 + \frac{x}{1!} + \frac{x}{2!} + \frac{x}{3!} + \dots + \frac{x^n}{n!}$ be denoted by P_n for brevity.

Observe that P_n has the following properties:

$$\text{(i)} \quad \frac{dP_n}{dx} = P_n^1 = P_{n-1}$$

$$\text{(ii)} \quad P_n = P_{n-1} + \frac{x^n}{n!}$$

1. $P_2 = \frac{1}{2}(x^2 + 2x + 2) = \frac{1}{2}[(x+1)^2 + 1] > 0$

for all x. Obviously P_2 has no real zeros.

2. Observe that $\frac{dP_3}{dx} = P_3' = P_2 > 0$ for all x. Since P_2 has no zeros, P_3 has no turning points, while the absolute value of the largest term is x^3 in P_3. This term is positive as $x \to \infty$ and negative as $x \to -\infty$, hence P_3 will cross the real axis at some point, i.e. it has one real root.

3. Observe that $\frac{dP_4}{dx} = P_4' = P_3$ and by (2) P_3 has just one zero, say at $x = a (a \neq 0)$.

$\frac{d^2P_4}{dx^2} = \frac{dP_3}{dx} = P_3' = P_2 > 0$, so P_4 has a minimum. Observe also that

$$P_4(a) = P_3(a) + \frac{a^4}{4!} > 0, \text{ since } P_3(a) = 0.$$

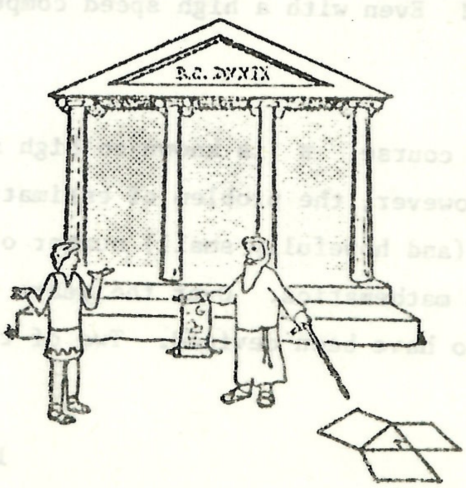
Hence $\frac{1}{24}a^4$ is the minimum value of P_4 and $P_4(x) \neq 0$ for any real value of x.

4. The proof for any n is by induction. Let N be an <u>even</u> integer, and assume that $P_N(x)$ has no real zero. Now $\frac{dP_{N+1}}{dx} = P_{N+1}' = P_N > 0$ for all real x, hence P_{N+1} has no turning points. It follows that P_{N+1} has just one real zero at, say, $x = a$. (Note that $P_{N+1}(0) = 1$, hence $a \neq 0$).

Consider now $\frac{dP_{N+2}}{dx} = P_{N+2}' = P_{N+1}$.

Since P_{N+1} has just one zero and $\frac{d^2P_{N+2}}{dx^2} = P_{N+1}' = P_N > 0$, it follows that P_{N+2} has just one turning point which is a minimum at $x = a$. Moreover as $P_{N+2} = P_{N+1} + \frac{x^{N+2}}{(N+2)!}$, it follows that $P_{N+2}(a) = \frac{a^{N+2}}{(N+2)!} > 0$. Hence $P_{N+2}(x) > 0$ for all real x.

But we have already shown that P_2, P_4 have no real zeros. Hence by induction $P_6, P_8, \ldots, P_N$ have no real zeros whenever N is an even integer.



THAT'S ALL VERY INTERESTING,
PYTHAGORAS — BUT WILL
IT HELP ME GET A JOB ?