SUMMING INFINITE SERIES

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Summing series of the form $u_1 + u_2 + \ldots + u_N$ is a common problem in mathematics. For some of these series it is possible to find a compact formula by algebraic methods, as for example for the geometric series you probably already know:

$$S_N = a + ar + \ldots + ar^{N-1} = \frac{a(1 - r^N)}{1 - r} . \qquad (1)$$

Furthermore you know that for $|r| < 1$ you can extend the summation to $N \to \infty$, so that

$$a \sum_{n=1}^{\infty} r^{n-1} = \frac{a}{1 - r} . \qquad (2)$$

In general we say that $\sum_{n=1}^{\infty} u_n = U$ has a limit sum if U is finite. The conditions under which such a limit exists are studied at university. Except for noting that an obvious necessary condition for U to exist is that $\lim_{N \to \infty} u_N = 0$, we will not discuss these conditions further in this article. Rather our interest will be in estimating the value of U (assumed to be finite) given the terms of the series $u_1, u_2, \ldots$ up to... u_N.

As a specific example we shall consider the series

$$\sum_{n=1}^{\infty} n^{-2} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots \qquad (3)$$

which can be shown to equal $\frac{\pi^2}{6} = 1.64493406\ldots$. Presumably we could utilise this series to estimate the value of π . One would obviously not obtain π exactly unless all terms are summed. The question is now: How many terms are needed to evaluate π to, say, eight significant figures? The answer is sixty million! Even with a high speed computer this is a daunting if not impossible task.

Of course, π is known to high accuracy from methods other than summing (3). However, the problem of estimating the sum of an infinite series from a finite (and hopefully small) number of its terms arises in many areas of applied mathematics. Over the years various methods, some very elaborate, of doing so have been devised. Two of the best methods are however based on nothing more than simple properties of the geometric series.

Let us write

$$U = \sum_{n=1}^{\infty} u_n = U_N + R_N , \qquad (4)$$

where we assume we know

$$U_N = \sum_{n=1}^{N} u_n = u_1 + u_2 + \ldots + u_N$$

and u_{N+1} but do not know the remainder R_N

$$R_N = \sum_{n=N+1}^{\infty} u_n = u_{N+1} + u_{N+2} + \ldots \qquad (6)$$

Obviously to estimate U we need to estimate R_N. A very simple way of doing this was suggested in the early 1950's by an American applied mathematician, David Shanks: regard R_N as a geometric series with initial term u_{N+1} and a common ratio given by $r_N = u_{N+1}/u_N$. (Since U is assumed to exist, we have $|r_N| < 1$.) Hence

$$R_N \simeq u_{N+1}/(1 - r_N) = u_{N+1}u_N/(u_N - u_{N+1}) \qquad (7)$$

and we estimate U by

$$U \simeq U_N^{(1)} = U_N + u_{N+1}u_N/(u_N - u_{N+1}) . \qquad (8)$$

As N increases we expect and can indeed prove that the new sequence $\{U_N^{(1)}\}$ has the same limit U as the original sequence. The hope is that $U_N^{(1)}$ is a better approximation than U_N.

Let us test this idea on the series (3). We have

$$U_1 = u_1 = 1 , \quad U_2 = u_1 + u_2 = 1\tfrac{1}{4} , \quad U_3 = u_1 + u_2 + u_3 = 1\tfrac{13}{36} = 1.361111\ldots \qquad (9)$$

so that simply summing the first three terms gives only one figure of the exact sum ($\pi^2/6 = 1.6449\ldots$). On the other hand, using (8) we find $U_1^{(1)} = U_1 + u_1u_2/(u_1 - u_2) = 1\frac{1}{3}$ and $U_2^{(1)} = 1\frac{9}{20} = 1.45$ which is significantly closer. The effect of incorporating further terms is shown in the second column of Table I. Note that given five terms in the original series, we obtain $U_1', U_2', \ldots, U_4'$.

We can now regard

$$U_N^{(1)} = \sum_{n=1}^{N} u_n^{(1)} = u_1^{(1)} + u_2^{(1)} + \ldots + u_N^{(1)} \tag{10}$$

where

$$u_1^{(1)} = U_1^{(1)}, \quad u_n^{(1)} = U_n^{(1)} - U_{n-1}^{(1)}, \quad n = 2,3,\ldots,N . \tag{11}$$

Hence we can repeat the transformation using (8) to define a second, a hopeful yet better, approximation to U namely

$$U_N^{(2)} = U_N^{(1)} + u_{N+1}^{(1)} u_N^{(1)} / \left(u_N^{(1)} - u_{N+1}^{(1)}\right) \tag{12}$$

which is shown in the third column of Table I. This in turn can be transformed until we build up the triangular pattern evident in Table I. Since each time we transform, or to use the technical term <u>accelerate</u>, the series, we lose one term, we can only apply (8) $N-1$ times if we have initially N terms. The striking result of Table I is that from five terms we estimate $\pi^2/6 \simeq 1.586$ whereas directly summing <u>ten</u> terms in (3) only gives $\pi^2/6 \simeq 1.55$.

We can do even better if we are a little more clever. Clearly we have made a very big assumption in approximating the remainder after N terms by a geometric series. Let us relax this assumption by allowing R_N to be not exactly $u_{N+1}u_N/(u_N - u_{N+1})$ but proportional to it, i.e. we write

$$R_N = \alpha u_{N+1} u_N / (u_N - u_{N+1}) . \tag{13}$$

The problem is now: what value do we use for α ? An answer to this can be found as follows. If we stop summing after N terms the remainder is R_N , while if we were to add one more term u_{N+1} to the sum we would be left with the remainder R_{N+1} . Clearly we should have

$$R_N = u_{N+1} + R_{N+1} . \tag{14}$$

Substituting (13) in (14) gives an equation for α , namely

$$\frac{\alpha u_{N+1} u_N}{u_N - u_{N+1}} = u_{N+1} + \frac{\alpha u_{N+2} u_{N+1}}{u_{N+1} - u_{N+2}} . \tag{14}$$

Solving for α we find

$$\alpha = \frac{(u_{N+1} - u_{N+2})(u_N - u_{N+1})}{u_{N+2}u_{N+1} + u_N u_{N+1} - 2u_N u_{N+2}} \tag{15}$$

so that

$$R_N = \frac{u_{N+1}u_N(u_{N+1} - u_{N+2})}{u_{N+2}u_{N+1} + u_N u_{N+1} - 2u_N u_{N+2}} . \tag{17}$$

Thus given $u_1, u_2, \ldots, u_{N+2}$, we now define a new approximation for the sum U by

$$U \approx \hat{U}_N^{(1)} = U_N + \frac{u_{N+1}u_N(u_{N+1} - u_{N+2})}{u_{N+2}u_{N+2} + u_N u_{N+1} - 2u_N u_{N+2}} . \tag{18}$$

The effect of this transformation, known as the θ-algorithm, on (3) is shown in the second column of Table II. Again we may successfully repeat the transformation to construct the rest of the table. Note that this time each transformation loses us two <u>terms</u>. Nevertheless, the table is rather spectacular; the third column gives $\pi^2/6$ to six figures. Indeed, if we use twenty terms in (3) we obtain

$$\frac{\pi^2}{6} = 1.6449340668431 \ldots$$

which is correct to twelve figures! A word of warning is appropriate if you are interested in trying these techniques for yourself. To obtain this accuracy one needs to work very precisely. If you keep only say six figures in your arithmetic you will only get the sum to probably at best three or four figures. It is best to try and keep all intermediate steps as fractions, only resorting to decimal numbers at the last moment. Nevertheless, you should be able to sum (3) to at least four figures using less than eight terms. Why don't you try? You might also like to try the following sums:

$$\sum_{n=1}^{\infty} \frac{(2n-1)}{n(n+1)(n+2)} = \frac{3}{4}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$$

Finally, if you think that this is all a bit mysterious you are not alone. The mathematical reasons why these sorts of transformations work so well are very unclear. Indeed, the whole area of accelerating the rate at which an infinite sum approaches its limiting value is a continuing area of research.

TABLE I: Estimation $\pi^2/6$ by Accelerating Series (3) Using Shank's Transform

n	u_N	U_N	$U_N^{(1)}$	$U_N^{(2)}$	$U_N^{(3)}$	$U_N^{(4)}$
1	1	1	1.3333	1.4612	1.5562	1.5858
2	$\frac{1}{4}$	1.25	1.4500	1.5504	1.5852	
3	$\frac{1}{9}$	1.3611	1.5040	1.5754		
4	$\frac{1}{16}$	1.4236	1.5347			
5	$\frac{1}{25}$	1.4636				

TABLE II: Estimation of $\pi^2/6$ by Accelerating Series (3) Using the θ- algorithm

n	U_N	$\hat{U}_N^{(1)}$	$\hat{U}_N^{(2)}$	$\hat{U}_N^{(3)}$	$\hat{U}_N^{(4)}$
1	1	1.625	1.645745	1.644921	1.644935
2	1.25	1.638888	1.644895	1.644938	
3	1.361111	1.642361	1.644923	1.644934	
4	1.423611	1.643611	1.644930		
5	1.463611	1.644167	1.644932		
6	1.491389	1.644450			
7	1.511797	1.644610			
8	1.527422				
9	1.539768				