PROBLEM SECTION

Several of the following problems (491-3, and 498-501) were amongst those proposed for this years School Mathematics Competition. Questions 496 and 497 were set in last years U.S. Olympiad. The others appeared in similar competitions in Sweden at various times.

MISPRINT Question 484 set in the last issue has a misprint in the last line, where the word "changed" should read "charged".

Q. 491. Find a four digit number which becomes nine times as large if the order of the digits is reversed. Find a five digit number with the same property. How many different 12 digit numbers have this property?

Q. 492. Find all positive integers, n, which have the following properties:-

a) the sum and the product of the (decimal) digits of n are both prime numbers;

and b) in whichever order we write down the digits of n, the resulting number will always be a prime number.

Q. 493. Two tetrahedra are congruent if corresponding edges are equal. Show that a cube may be cut into six congruent tetrahedra.

Q. 494. Find all real numbers x which satisfy the equation:-

$$\sqrt{x^2} + \sqrt{(x-1)^2} = 1.$$

Q. 495. Let n be any odd number, and let $a_1, a_2, a_3, \ldots, a_n$ be any rearrangement of the numbers $1,2,3,\ldots, n$. Prove that the product $(a_1-1)(a_2-2)(a_3-3)\ldots(a_n-n)$ is even.

Q. 496. A two-pan balance is inaccurate since its balance arms are of different lengths and its pans are of different weights. Three objects of different weights A,B and C are each weighed separately. When placed on the left hand pan, they are balanced by weights A_1, B_1, C_1 respectively. When A and B are placed on the right hand pan they are balanced by A_2 and B_2 respectively. Determine the true weight of C in terms of A_1, B_1, C_1, A_2 and B_2.

Q. 497. The inscribed sphere of a given tetrahedron touches each of the four faces of the tetrahedron at their respective centroids. Prove that the tetrahedron is regular (i.e. all six edges are of equal length).

Q. 498. Solve the system of equations

$$x + y^2 + z^3 = 3$$
$$y + z^2 + x^3 = 3$$
$$z + x^2 + y^3 = 3$$

in positive numbers x,y,z.

Show that there are no solutions if all three numbers are negative.

Q. 499. A salesman works for a firm which sells 99 different lines of merchandise. Each day from Monday to Saturday (inclusive), he packs fifty different lines into his sample case. Show that there is at least one pair of lines such that on every one of the six days at least one of the pair was in the sample case.

Q. 500. A mixed school organizes a chess competition in which every participant plays one game against every one else. At the conclusion of the tournament, it is noted that each competitor has gained exactly half of his or her points in games against boys. (As usual, a win earns 1 point, and a draw earns ½ point for both players). Prove that the total number of competitors is a perfect square.

Q. 501. The numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ in an infinite list satisfy $a_1 \geq 2$, $a_{n+1} = a_n^2 - 2$ $(n=1,2,\ldots)$. Show that the sum of the series

$$\frac{1}{a_1} + \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2 a_3} + \ldots$$

is $\frac{a_1}{2} - \sqrt{\left(\frac{a_1}{2}\right)^2 - 1}$.

Q. 502. Find all real polynomials P and Q such that for every real number a, the number $P(a)$ satisfies the equation

$$x^3 + Q(a)\,x^2 + (a^4+1)\,x + a^3 + a = 0.$$