SOLUTIONS TO PROBLEMS FROM VOLUME 16, NUMBER 3.

Q. 467. In a plane are 127 toothed cog-wheels, numbered (1), (2), ..., (127). The teeth of wheel (1) engage those of wheel (2), and similarly (2) is engaged with (3), (3) with (4), and finally (127) with (1). Can the cogwheels of the system be turned? Justify your assertion.

SOLUTION: Disengage wheel (1) from wheel (127). If (1) is now turned anticlockwise, (2) is forced to turn clockwise, (3) anticlockwise, and so on. In fact wheel (2n) must turn clockwise and (2n+1) anticlockwise for n = 1,2,...63. Thus (127) turns anticlockwise. If we now re-engage (127) with (1), an anticlockwise motion of (127) would force (1) to turn clockwise. Since it is impossible for (1) to rotate simultaneously in both directions, no motion at all is possible.

Correct Solutions from: D. Everett (Kotara High School); A. Cherry; R. Bozier; K. Lim (St. Ignatius College); M. Huesch (Forster High School).

Q. 468. Find a rapid way of calculating $(999,999,999)^3$, and perform the calculation.

SOLUTION: Substituting 10^9 for A and 1 for B in the identity

$(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$ gives the answer

$10^{27} - 3.10^{18} + 3.10^9 - 1 = (10^9-3).10^{18} + 2,999,999,999$

$= 999,999,997,000,000,002,999,999,999$.

Correct Solutions from: D. Everett (Kotara High School); A. Cherry; R. Bozier; K. Lim (St. Ignatius College); M. Huesch (Forster High School).

Q. 469. Solve the following equation

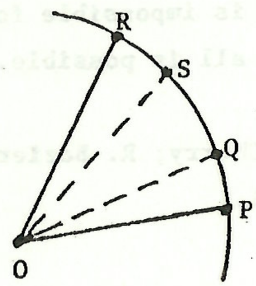
$$\cos x + \cos^5 x + \cos 7x = 3.$$

SOLUTION: Since $\cos\theta \le 1$, the equation can be satisfied only if every term on the L.H.S. is equal to 1. The first term, cos x, equals 1 if and only if x is a multiple of 2π (or 360°). For all these values of x, $\cos^5 x$ and cos 7x are equal to 1, and the equation is satisfied. Hence all solutions are given by $x = 2k\pi$ where k is an integer $k = 0, \pm1, \pm2, \pm3, ..$, [or x = 360 k if you understood x to be the measure of an angle in degrees].

Correct Solutions from: D. Everett (Kotara High School); K. Lim (St. Ignatius College); M. Huesch (Forster High School).

Q. 470. In a circle centre O are inscribed a regular polygon of n sides, and a regular polygon of (n+1) sides. Show that it is possible to choose a vertex P of the first polygon and a vertex Q of the second such that the angle POQ is less than $\frac{360}{n(n+1)}$ degrees.

SOLUTION:



The vertices of the n-gon divide the circle into n equal arcs, on which are situated the (n+1) vertices of the (n+1)-gon. Hence some arc PR, terminated by 2 vertices of the n-gon must contain 2 vertices QS of the (n+1)-gon. Then

$$\left(\frac{360}{n}\right)^\circ = \hat{POR} = \hat{POQ} + \hat{QOS} + \hat{SOR} = \hat{POQ} + \left(\frac{360}{n+1}\right)^\circ + \hat{SOR}.$$

$$\therefore \hat{POQ} = \left(\frac{360}{n} - \frac{360}{n+1}\right)^\circ - \hat{SOR} < \left(\frac{360}{n(n+1)}\right)^\circ.$$

[Indeed, the smaller of $\hat{POQ}$ and $\hat{SOR}$ is $\leqslant \left(\frac{180}{n(n+1)}\right)^\circ$.

Correct Solution from: K. Lim (St. Ignatius College).

Q. 471. A triangle lies inside a polygon. Prove rigorously that the perimeter of the triangle is less than the perimeter of the polygon.

SOLUTION:

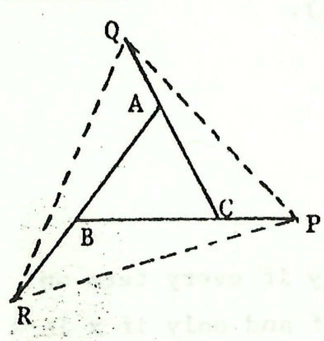


Fig. 1

Produce BC until it intersects the polygon at P. Similarly let Q,R be points on the polygon lying on CA produced and AB produced respectively. Join PQ, QR, RP (see fig. 1). The points PQR divide the perimeter of the polygon into three pieces. The piece joining P to Q has a length not less than the straight line segment PQ. Similarly for the other two pieces. Hence, adding

Perimeter of polygon ≥ PQ* + QR* + RP* (1)

Since PQ* + CP* > CQ* = CA* + AQ*

and QR* + AQ* > AR* = AB* + BR*

and RP* + BR* > BP* = BC* + CP*

adding and deleting AQ* + BR* + CP* from both sides gives

$$PQ^* + QR^* + RP^* > AB^* + BC^* + CA^* \qquad (2)$$

From (1) and (2) we have

Perimeter of polygon > perimeter of triangle.

Correct Solution from: K. Lim (St. Ignatius College).

Q. 472. Show that any three real numbers x, y, z which satisfies

$$\sin x + \sin y + \sin z = 0$$

$$\cos x + \cos y + \cos z = 0$$

also satisfy

$$\sin 2x + \sin 2y + \sin 2z = 0$$

$$\cos 2x + \cos 2y + \cos 2z = 0.$$

SOLUTION: We choose an indirect proof based on the following facts:-

(1) The centroid of a triangle with vertices (x_1,y_1), (x_2,y_2) and (x_3,y_3) is the point $(\frac{1}{3}(x_1+x_2+y_3), \frac{1}{3}(y_1+y_2+y_3))$.

(2) The centroid is the intersection of the medians.

(3) The circumcentre is the intersection of the perpendicular bisectors of the sides.

To start, observe that if $x - x'$ is a multiple of 2π, where $0 < x' < 2\pi$, then $\cos x = \cos x'$, $\sin x = \sin x'$, $\cos 2x = \cos 2x'$, and $\sin 2x = \sin 2x'$. Using this it follows that we can assume that x lies in the range $[0,2\pi)$, and similarly for y and z. By the symmetry of the expressions, we can also assume without loss of generality that $x \leq y \leq z$. Now plot the points X, Y, Z on the unit circle in the Cartesian plane, making angles of x, y, z respectively with the horizontal axis. Their co-ordinates are X(cos x, sin x), Y(cos y, sin y) and Z(cos z, sin z). Now O, the origin, is not only the circumcentre of ΔXYZ, but in view of the given equations, and (1) above, it is also the centroid of the triangle. It follows that the median from X is the perpendicular bisector of YZ, whence XY = XZ. Similarly XY = YZ, so ΔXYZ is equilateral. Thus $X\hat{O}Y = Y\hat{O}Z = Z\hat{O}X = \frac{2\pi}{3}$ i.e. $y - x = z - y = \frac{2\pi}{3}$. Doubling, we obtain

$$2y - 2x = 2z - 2y = \frac{4\pi}{3}. \qquad (4)$$

If P, Q, R are points on the unit circle such that OP, OQ and OR make angles of

2x, 2y and 2z respectively with the horizontal axis, then ΔPQR is equilateral because of (4). Hence its circumcentre, O, is the same point as its centroid. Now, using (1) again, we deduce that

$$\cos 2x + \cos 2y + \cos 2z = 0 \text{ and } \sin 2x = \sin 2y + \sin 2z = 0,$$

as required.

Q. 473. Prove that the equation $x^n + y^n = z^n$ (n an integer > 1) has no solution in integers x, y, z with $0 < x \leqslant n$, $0 < y \leqslant n$.

SOLUTION: We can assume that $x \leqslant y$. Then, since z is an integer greater than y,

$$z^n \geqslant (y+1)^n = y^n + ny^{n-1} + {}^nC_2 y^{n-2} + \ldots$$

$$> y^n + ny^{n-1} \geqslant y^n + y^n \text{ since } n \geqslant y$$

$$\geqslant x^n + y^n \text{ since } y \geqslant x$$

$$\therefore z^n > x^n + y^n .$$

Q. 474. Find a set S of 7 consecutive positive integers for which there exists a polynomial P(x) of the fifth degree with the following properties:

(a) all the coefficients of P(x) are integers

(b) P(n) = n for five members, n, of S, including the least and greatest

(c) P(n) = 0 for one member of S.

SOLUTION: Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 be five members of S such that P(x) = x. By the factor theorem $P(x) - x = c(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5)$ where c is an integer. Let α_6 be the member of S such that $P(\alpha_6) = 0$. Then $-\alpha_6 = c(\alpha_6-\alpha_1)(\alpha_6-\alpha_2)(\alpha_6-\alpha_3)(\alpha_6-\alpha_4)(\alpha_6-\alpha_5)$. The factors on the RHS are all integers between −5 and +5, the largest and smallest differing by 6. There are infinitely many solutions. For one take c = 1, $-\alpha_6 = 1.-4.-2.-1.1.2$. This gives $\alpha_6 = 16$, $\alpha_1 = 20$, $\alpha_2 = 18$, $\alpha_3 = 17$, $\alpha_4 = 15$, $\alpha_5 = 14$. The polynomial is

$$P(x) = x + (x-20)(x-18)(x-17)(x-15)(x-14)$$

and $S = \{14,15,16,17,18,19,20\}$.

Q. 475. On the diameter AB bounding a semicircular region, there are two points P and Q, and on the semicircular arc there are two points R and S, such that PQRS is a square. C is a point on the semicircular arc such that the areas of the triangle ABC and the square PQRS are equal. Prove that a straight line passing through one of the points R and S, and through one of A and B, cuts a side of the square at the in-centre of the triangle.

SOLUTION:

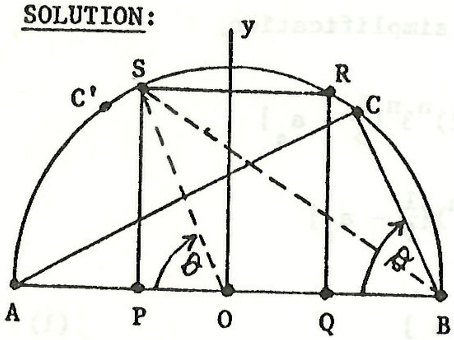


Fig. 1

We shall prove that, as we have drawn the figure, BS bisects $\hat{ABC}$, and hence contains the in-centre of ΔABC. From the isosceles ΔOBS $\hat{OBS} = \hat{OSB} = \hat{AOS}$. Hence we will show that $\hat{SBO} = \frac{1}{2}\hat{ABC}$ if we show that $\hat{ABC} = \hat{AOS}$. (1)

Since PQRS is a square $OP^* = \frac{1}{2} PQ^* = \frac{1}{2} PS^*$.

$\therefore \tan \hat{POS} = PS^*/OP^* = 2$. $\therefore \sin \hat{POS} = \frac{2}{\sqrt{5}}$

$\therefore$ Area of square $= (PS^*)^2 = (r \sin \hat{POS})^2$

$$= r^2 \frac{4}{5} \qquad (2)$$

where r is the radius of the circle.

$$\therefore \text{Area } \Delta ABC = \tfrac{1}{2} AB.BC \sin \beta$$
$$= \tfrac{1}{2}.2r.(2r \cos \beta) \sin \beta$$
$$= r^2 \sin 2\beta \qquad (3)$$

Equating (2) and (3) yields $\sin 2\beta = \frac{4}{5}$

$$\therefore \sin 2\beta = \frac{2 \tan \beta}{1+\tan^2\beta} = \frac{4}{5} \qquad \text{which gives}$$

$$2 \tan^2\beta - 5 \tan \beta + 2 = 0$$

$$(2 \tan \beta - 1)(\tan \beta - 2) = 0$$

$$\tan \beta = \tfrac{1}{2} \text{ or } \tan \beta = 2.$$

Since $\beta > \frac{\pi}{4}$, we must take the value $\tan \beta = 2$. (The other value $\tan \beta = \frac{1}{2}$ would give the triangle ABC', obtained by reflecting the figure about Oy). Since $\tan \hat{ABC} = 2 = \tan \hat{POS}$ we have accomplished our objective i.e. $\hat{ABC} = \hat{AOS}$.

Q. 476. A list of real numbers $a_0, a_1, a_2, \ldots, a_n, \ldots$ has the following properties:

(a) $a_{n+1} = 2^n - 3a_n \quad (n \geq 0)$

(b) $a_{n+1} > a_n$ $(n \geq 0)$.

Prove that there is just one such list, and find a_0.

SOLUTION: Using (a) repeatedly, one arrives easily at the formula

$$a_n = 2^{n-1} - 2^{n-2}\,3 + 2^{n-3}.3^2 - \ldots + (-1)^{n-1}3^{n-1} + (-1)^n 3^n a_0$$

(which can be formally proved by induction if desired). The formula for summing a G.P. with ratio $-\frac{3}{2}$ now gives, after a little simplification,

$$a_n = \frac{2^n-(-3)^n}{5} + (-1)^n 3^n a_0 = \frac{2^n}{5} - (-1)^n 3^n [\tfrac{1}{5} - a_0]$$

Hence $$a_{n+1} - a_n = \frac{2^{n+1}-2^n}{5} + (-1)^n(3^{n+1}+3^n)[\tfrac{1}{5} - a_0]$$

$$\frac{a_{n+1} - a_n}{3^n} = \left(\frac{2}{3}\right)^n \frac{1}{5} + (-1)^n.4.[\tfrac{1}{5} - a_0] \qquad (1)$$

Now if $a_0 = \frac{1}{5}$, $a_{n+1} - a_n = \frac{2^n}{5} > 0$ for all n so that (b) is satisfied. However for any other value of a_0, the second term on the R.H.S. of (1) has a constant absolute value as n increases, but is alternately positive and negative; while the first term becomes negligibly small for sufficiently large values of n. Hence (b) is not satisfied for any value of a_0 except $a_0 = \frac{1}{5}$.

Correct Solution from: D. Everett (Kotara High School).

Q. 477. In a party of ten persons, among any three persons there are at least two who do not know each other. Prove that in the party there are four persons none of whom knows another of the four.

SOLUTION: Let one of the party, A say, be acquainted with the people in the set $\chi_A = \{X_1, X_2, \ldots, X_k\}$. No X_i is acquainted with an X_j in χ_A, since the three people A, X_i, X_j would contradict the data. Thus if $k \geq 4$ $\{X_1, X_2, X_3, X_4\}$ would be a set of 4 mutually unacquainted people. Henceforth we may assume that no one at the party knows more than three others.

On the other hand, if $k < 3$, there are $10-k-1 \geq 7$ people not known by A. Let B be one of them. Since B knows at most 3 of the seven there are at least 3, {X,Y,Z, say} who are not acquainted with B. Of these 3 two {X and Y, say} are unacquainted, by the data. Then {A,B,X,Y} is the required foursome.

We have only one case remaining:- that in which each person knows precisely 3 of the others. Unfortunately this is the most complicated case to resolve. Let A be acquainted with all members of the set $S \equiv \{B,C,D\}$ and therefore unacquainted with all members of the set $T \equiv \{E,F,G,H,I,J\}$. If any of the latter, E say, were unacquainted with B,C and D then the foursome $\{E,B,C,D\}$ would terminate our search. Hence each of B,C and D must be acquainted with 2 different people in the set T. We can relabel the members of T so that $T \equiv \{E_B, F_B, G_C, H_C, I_D, J_D\}$ where the subscript indicates the member of S with which each is acquainted.

We now turn our attention to friendships within T. No two bearing the same subscript can be acquainted, by the data.

We shall connect the points in Fig. 1 by lines to indicate that the corresponding people are acquainted. Altogether there are 6 lines to be put in, two from each vertex. If any of G,H,I or J is unacquainted with both E and F, then we finish by exhibiting the foursome comprising that person with A,E and F. Thus we may assume that the four lines starting from E and F must finish one each at G,H,I and J. Similarly the four lines from G and H must be distributed evenly amongst F,E,I and J; etc.

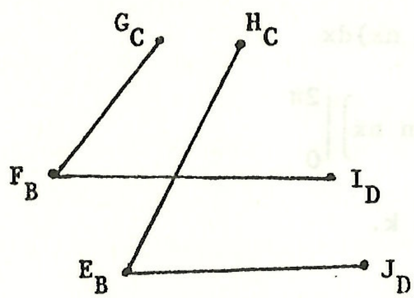


Fig. 1

Hence, relabelling within the pairs (G,H) and/or (I,J) if necessary, we can assume that E_B is joined to H_C and J_D, and that F_B is joined to G_C and I_D. To avoid the triangle $G_C F_B I_D$, we must put in the remaining 2 lines by joining I_D to H_C and J_D to G_C. But then $\{A, E_B, G_C, I_D\}$ has the required properties. This completes the argument.

<u>Q. 478.</u> Decide whether the following statement is true or false:- There exists a number N such that every natural number greater than N is the sum of two fourth powers of integers i.e. if $Z > N$ then there exists integers x,y such that $x^4 + y^4 = Z$.

<u>SOLUTION</u>: False. For example, the last digit of x is 0,1,6,1,6,5,6,1, 6 or 1 according as the last digit of x^4 is 0,1,2,3,4,5,6,7,8 or 9. Hence the last digit of $x^4 + y^4$ is 0,1,2,5,6 or 7; but no number ending in 3,4,8 or 9 is so

expressible.

Correct Solutions from: D. Everett (Kotara High School), K. Lim (St. Ignatius College).

Q. 479. Decide whether the following statement is true or false:- There exists numbers $a_1, a_2, \ldots, a_n$ such that

$$a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx > 0 \quad \text{for all } x.$$

SOLUTION: False again. If $F(x) > 0$ for all x then $\int_0^{2\pi} F(x)dx > 0$. But

$$\int_0^{2\pi} (a_1 \cos x + a_2 \cos 2x + \ldots + a_n \cos nx)dx$$

$$= \left[-a_1 \sin x - \frac{a_2}{2} \sin 2x - \ldots - \frac{a_n}{n} \sin nx\right]_0^{2\pi}$$

$$= 0 \quad \text{since } \sin k\pi = 0 \text{ for all integers } k.$$

Q. 480. In the Pascal triangle the nth row consists of the integers $1, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$. Let P_n denote their product. Prove that

$$\frac{P_n}{P_{n-1}} = \frac{n^{n-1}}{(n-1)!}.$$

SOLUTION:

$$\frac{P_n}{P_{n-1}} = \frac{1.{}^nC_1.{}^nC_2 \ldots\ldots\ldots {}^nC_{n-1} \quad {}^nC_n}{1.{}^{n-1}C_1.{}^{n-1}C_2 \ldots\ldots\ldots {}^{n-1}C_{n-1}}.$$

Note that $\frac{{}^nC_r}{{}^{n-1}C_r} = \frac{n(n-1)\ldots(n-r+1)}{(n-1)(n-2)\ldots(n-r)} = \frac{n}{n-r}$.

Hence $\frac{P_n}{P_{n-1}} = \frac{n}{n-1} \cdot \frac{n}{n-2} \ldots\ldots\ldots\ldots\ldots \frac{n}{(n-(n-1))}$

$$= \frac{n^{n-1}}{(n-1)!}.$$