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# HEAT CONDUCTION IN A SPHERE
OR
# "HOW LONG TO COOK A ROAST?"

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On several occasions I have been in the position of having to cook a roast dinner. My experience in this field is rather limited and so I have to rely on cookbooks to provide cooking temperature and time. Let me quote from two cookbooks:

"Bake in moderately hot oven.... Allow approximately 20 to 30 minutes per 500 g (lb) cooking time, depending on thickness...." (1)
and "medium to well done - 20 minutes per lb and 20 minutes over." (2)

Cooking time for the roast (beef in both cases) as a function of the weight is shown sketched in Fig. 1., and from both books we have a <u>linear</u>

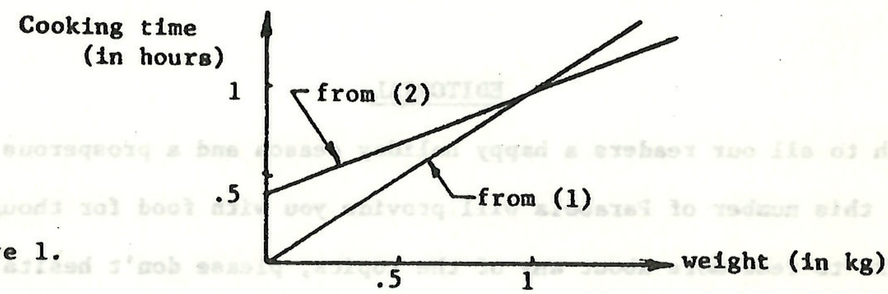


Figure 1.

relationship between cooking time and weight. But which one, if any, of these two functions gives the correct cooking time for a given roast? I would like to demonstrate, using a little mathematics, that neither can be correct, as the relationship between cooking time and weight is not linear.

First of all, what does cooking involve? It involves heating food to a sufficiently high temperature so that the process which turns uncooked food into the cooked state can take place; let $T^*$ denote the temperature at which this reaction takes place. Next we realise that the shape of the food to be cooked is important. Thinly sliced food is cooked in a much shorter time than the same mass of stuff in one solid lump. The Chinese exploit this fact superbly in their cooking, whilst Europeans waste time and heat by using the slow process of heat conduction to raise the temperature of the interior of the food to $T^*$.

The shape of most roasts is not very regular, but to get a quick answer, let us consider a <u>spherical</u> roast! As an approximation to the shape of a leg of lamb a sphere is terrible, not too bad for rolled sirloin or pork, O.K. for potatoes and quite good for onions or apples. Clearly, the centre of the spherical roast will take the longest time to reach temperature $T^*$, so our problem can now be stated as follows: "How long does it take for the centre of a sphere to reach temperature $T^*$ if the oven is at temperature $T_o$, and initially the sphere had uniform temperature $T_i$?" In the context of cooking a roast the temperatures are such that

$$T_i < T^* < T_o$$

The problem we have posed is very familiar to applied mathematicians, and is also of considerable importance to engineers, who refer to it as "the dunking problem". Before answering the question, let us use some physical intuition to work out the temperature variation inside the sphere (of radius a). Let $r$ denote the distance from the centre of the sphere, and let $t$ be the time since the roast was put in the oven. Then, at any <u>fixed</u> value of $r$ we expect the temperature to increase from $T_i$ up to $T_o$ as shown in Fig. 2(a); if left in sufficiently long (actually for an infinite length of time) the whole sphere

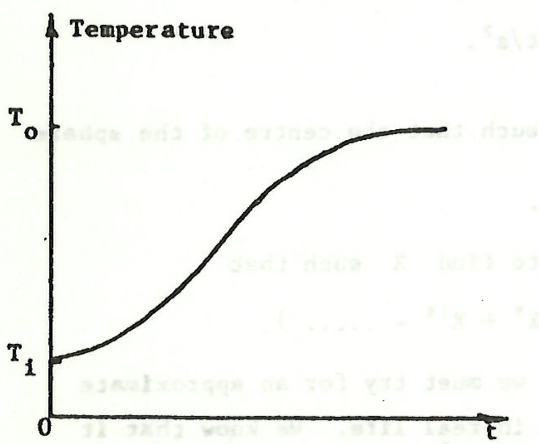


Figure 2(a)

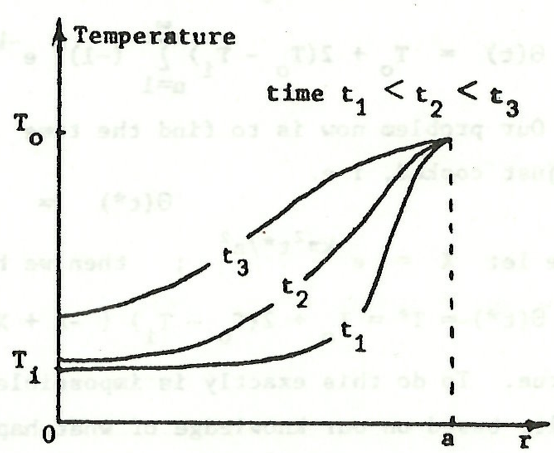


Figure 2(b)

would attain temperature $T_o$. Alternatively, we can look at the temperature variation as a function of $r$ for fixed values of time $t$. Clearly, the outer edges of the sphere will be the hottest, with the temperature decreasing as we get closer to $r = 0$. Fig. 2(b) shows the expected temperature variation for different times. Note that I have assumed that the surface of the sphere, $r = a$, reaches temperature $T_o$ instantaneously, and remains at that temperature for all $t > 0$.

Unfortunately, the temperature as a function of $r$ and $t$, denoted by $T(r,t)$, which satisfies the equations of heat conduction, cannot be derived by elementary methods, so let me quote, direct from the standard book on heat conduction by Carslaw and Jaeger (3). The temperature in the sphere is

$$T(r,t) = T_o + 2(T_o - T_i) \sum_{n=1}^{\infty} (-1)^n e^{-kn^2\pi^2 t/a^2} \frac{\sin(n\pi r/a)}{(n\pi r/a)}$$

$$\text{for} \quad 0 < r \leq a \quad \text{and} \quad t > 0.$$

The constant $k$ appearing in the exponential function is called the "thermal diffusivity" of the material. Good conductors of heat, such as metals, have a large value for $k$, poor conductors of heat, such as air, water and food, have a small value for $k$.

Now we would like the temperature at the centre of the sphere, but due to the $r$ in the denominator we cannot just set $r = 0$ in the expression for $T(r,t)$. Instead, we note that the numerator contains $\sin\left(\frac{n\pi r}{a}\right)$, and so we take the limit as $r \to 0$, using the facts that

$$\lim_{r \to 0} \frac{\sin\left(\frac{n\pi r}{a}\right)}{(n\pi r/a)} = \lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \text{using the substitution } x = \frac{n\pi r}{a}.$$

Now let us denote $\lim_{r \to 0} T(r,t)$ by the function $\Theta(t)$, where

$$\Theta(t) = T_o + 2(T_o - T_i) \sum_{n=1}^{\infty} (-1)^n e^{-kn^2\pi^2 t/a^2}.$$

Our problem now is to find the time $t^*$ such that the centre of the sphere has just cooked, i e.

$$\Theta(t^*) = T^* .$$

If we let $X = e^{-k\pi^2 t^*/a^2}$; then we have to find $X$ such that

$$\Theta(t^*) = T^* = T_o + 2(T_o - T_i) \{ -X + X^4 - X^9 + X^{16} - \ldots.. \}$$

is true. To do this exactly is impossible, so we must try for an approximate result, based on our knowledge of what happens in real life. We know that it takes a long time to cook a roast, and as $e^{-k\pi^2 t^*/a^2}$ decreases monotonically from unity, it is reasonable, as a first guess, to assume that $X$ is very much less than one. In which case $X^4$, $X^9$, $X^{16}$ etc. are even smaller still and can be neglected, compared to $X$. This approximation can be checked for consistency, after we have found an approximate solution. Thus neglecting the powers of $X$ in the above equation we get the approximate result

$$X = \frac{1}{2}\,\frac{T_o - T^*}{T_o - T_i}\,,$$ and recalling the

definition of X, and taking the natural logarithm of both sides we get

$$-k\pi^2 t^*/a^2 = \ell n \left\{ \frac{1}{2}\,\frac{T_o - T^*}{T_o - T_i} \right\}, \text{ or}$$

after some manipulations,

$$t^* = \frac{a^2}{k\pi^2}\,\ell n \left\{ 2\,\frac{T_o - T_i}{T_o - T^*} \right\},$$

But we want cooking time $t^*$ as a function of the weight $W$ of the roast. As we have a spherical roast, the weight is

$$W = \frac{4}{3}\pi a^3.\rho \quad \text{where } \rho \text{ is the density}$$

of the roast. Using this to eliminate $a^2$ from the previous result, we finally conclude that

$$t^* = W^{2/3}.\left(\frac{3}{4\pi\rho}\right)^{2/3}.\frac{1}{k\pi^2}\,\ell n \left\{ 2\,\frac{T_o - T_i}{T_o - T^*} \right\},$$

or, in words, the cooking time is proportional to the $^2/_3$ power of the weight of the roast!! The linear relationship given by the cookbooks has no mathematical foundation, and at best must be regarded as an approximation to the $^2/_3$ power law we have just derived. This probably explains why different cookbooks give different cooking times.

One last thing before finishing - we must check that our approximation of small $X$ is valid, and to do this we need some values for $T_o$, $T_i$ and $T^*$. For roasting meat, the oven temperature is about 180°C and we can take the initial temperature of the roast to be room temperature, 20°C. The temperature at which meat cooks, $T^*$, must be less than 100°C as we can cook meat by boiling it in water (i.e. stewing it), thus

$$X \simeq \frac{1}{2}.\frac{180-100}{180-20} = \frac{1}{4}\,, \quad \text{in which case}$$

$X^4 \simeq 0.004$ which is small enough to neglect, compared to $X$, given the other approximation made.

References


(1) "The Australian Womens Weekly Cookbook".

(2) Marguerite Patten, "The Australian Way to Perfect Cooking".

(3) Carslaw, H.S. and Jaeger, J.C. (1959) Conduction of heat in solids, Oxford University Press.