ANIMAL OLYMPICS

"Size of body is no mere accident. Man, respiring as he does, cannot be as small as an insect, nor vice versa... In fact each main group of animals has its mean and characteristic size..."

D'Arcy Wentworth Thompson

(On Growth and Form)

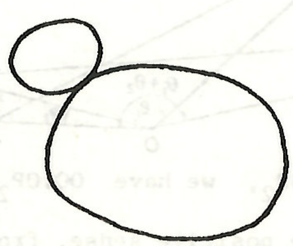
§1. All creatures great and small (The principle of similarity). We start by observing that if a cube has equal sides of length ℓ, then the area of each face is ℓ^2, its total surface area is $6\ell^2$, and its volume is ℓ^3. More generally if we consider a rectangular solid with edges of length $\ell, \beta\ell$ and $\gamma\ell$, then its surface area is $2(\beta + \gamma + \beta\gamma)\ell^2$ and its volume is $\beta\gamma\ell^3$. On the other hand a sphere with radius ℓ has surface area $4\pi\ell^2$ and volume $\frac{4}{3}\pi\ell^3$. For each of these three bodies we see that surface area A and volume V are given by

$$A = c\ell^2$$
$$V = d\ell^3$$

where c and d are constants and where ℓ is termed the linear dimension of the body. When such equations hold, we say that A is proportional to ℓ^2 and V is proportional to ℓ^3, and write

$$A \propto \ell^2$$
$$V \propto \ell^3.$$

In this article we shall make the "primitive art" assumption that an animal can be described by a linear dimension ℓ, that its surface area (or that of a limb) is proportional to ℓ^2, and that its volume (or, again, that of a limb) is proportional to ℓ^3. The astute reader will now be able to draw a cat, dog, horse, elephant etc. as follows



The principle of similarity can now be expressed: if two animals have linear dimensions ℓ and $k\ell$ (for some constant k) then their surface areas are proportional to ℓ^2 and $k^2\ell^2$ respectively and their volumes are proportional to ℓ^3 and $k^3\ell^3$ respectively. For example, if we take ℓ to measure head-and-body length in metres, then a red kangaroo is about 4 times larger than a potoroo ($k = 4$) and thus the red kangaroo has 16 times the hide, and 64 times the appetite, of a potoroo.

The principle of similarity (here a consequence of the primitive art assumption) was first observed by Archimedes on one of his few forays from the bath, and is usually stated with the assumption that the two animals be geometrically similar. (The trouble with this is that it is possible to deduce, by arguing as below, a contradiction that the animals are not geometrically similar!)

For another application of the principle of similarity, when Gulliver travels in Lilliput, we read: "His Majesty's Ministers, finding that Gulliver's stature exceeded theirs in the proportion of twelve to one, concluded from the similarity of their bodies that his must contain at least 1,728 of theirs, and must needs be rationed accordingly".

From the principle of similarity we can understand why the limbs of a mouse are much more delicate than the relatively massive limbs of an elephant. If ℓ denotes linear dimension, the animal's weight is proportional to ℓ^3, whilst the ability of its legs to support this weight is proportional to their cross sectional area, that is, to ℓ^2. So if an elephant were geometrically similar to a mouse, its legs would collapse under its own weight. Similarly, if Gulliver really were similar to the Lilliputians, the stress on his legs would be 12 times as large as that on a Lilliputian's, and so his bones would need to be 12 times as strong - an unlikely event. The more probable conclusion, that his legs would be $12^{1/2}$ times as thick, contradicts geometrical similarity.

§2. <u>Thigh bone's connected to the hip bone (Animal power)</u>. An animal's muscles convert chemical energy into mechanical energy and so the <u>power</u> an animal can generate is the power its muscles are physically capable of, limited by the rate at which oxygen can be supplied by its heart and by the rate at which heat can be dissipated.

We now see how each of these three, hence also power, is proportional to ℓ^2 and not, as might at first be thought, to ℓ^3.

Work done by a muscle in contracting is its force time distance moved. Muscle force depends on the number of fibrils able to contract, hence on cross sectional area, hence on ℓ^2 ; and distance depends on ℓ . So work done varies with ℓ^3 and provides a change in kinetic energy proportional to mass times speed squared, that is, to $\ell^3 v^2$. Hence v^2 is independent of ℓ so that distance over time is constant so time varies directly with ℓ. Thus power, which is proportional to work over time, varies with ℓ^2.

Next, the amount of oxygen available is determined by the amount of blood pumped out by the heart, which in turn is determined by the heart's power and is limited by the cross sectional area of the aorta, that is, to ℓ^2 (and not to the volume of the heart!).

Finally the rate at which heat can be dissipated is proportional to the animal's surface area, that is, to ℓ^2 . This last factor cannot be ignored since about 75% of the muscle energy takes the form of heat.

As a consequence we note that the volume of blood produced by the heart per unit of time varies with ℓ^2. On the other hand volume of blood per unit of time equals pulse rate times volume of blood to be pumped, hence

$$\text{pulse rate} \quad \alpha \quad \ell^2/\ell^3 \quad \alpha \quad \ell^{-1}.$$

Indeed mice have a far higher pulse rate than whales.

As another consequence, the larger a shark is, the faster it can get you (provided you don't cheat and stay in the shallows). For its speed depends on the work done by its muscles (proportional to ℓ^3), and is hampered by water resistance which is proportional to surface area, that is, to ℓ^2. Thus speed $\alpha \; \ell^3/\ell^2 = \ell$. For ships, this is called Froude's law.

§3. They're off and running, jumping, diving, horseriding. A horse will stop to walk uphill when a dog continues running; why? Power available to each is proportional to ℓ^2. The rate of increase in height uphill is proportional to speed v, so work done against gravity is mass times v, i.e. proportional to $\ell^3 v$. At maximum speed,

$$\ell^2 \propto \ell^3 v$$

$$\text{i.e.} \quad v \propto \ell^{-1} .$$

A jerboa can jump about as high as a red kangaroo; why? Work done by the "jumping muscles" equals the product of their downwards force on the ground (proportional to mass, i.e. to ℓ^3) and the height h jumped. Hence $h \propto \ell^3/\ell^3$ is independent of ℓ .

Whales can dive for far longer periods than man; why? Both mammals need to carry their oxygen from the surface, and lung volume available for this is proportional to ℓ^3 . The rate of consumption of oxygen varies with lung area, that is, with ℓ^2. Hence dive time is proportional to ℓ .

Why is there an upper limit (about 15 kilos) to the weight of animals with the ability to hover (not just to glide)? An animal hovers by producing a flow of downwards air to balance its weight (proportional to ℓ^3). If the wings have area A and speed v, the mass of air pushed downwards in unit time is proportional to A.v , so has momentum $Av^2 \propto \ell^2 v^2$. Thus,

$$\ell^3 \propto \ell^2 v^2$$

$$\text{or} \qquad v \propto \ell^{\frac{1}{2}} .$$

Work done by the wings is proportional to the kinetic energy of the air stream, i.e. to

$$Av.v^2 \propto \ell^2.\ell^{\frac{3}{2}} \propto \ell^{3.5}$$

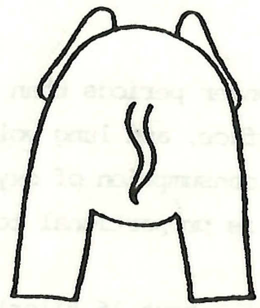
but is only available at a rate of ℓ^2 . Hence some upper limit must be reached to balance these.

There is no consistent variation with size of top running speed (for animals ranging from a cat to an elephant); why? Air resistance is proportional, it turns out, to surface area times speed v, squared, and the power for this is proportional to resistance times v :

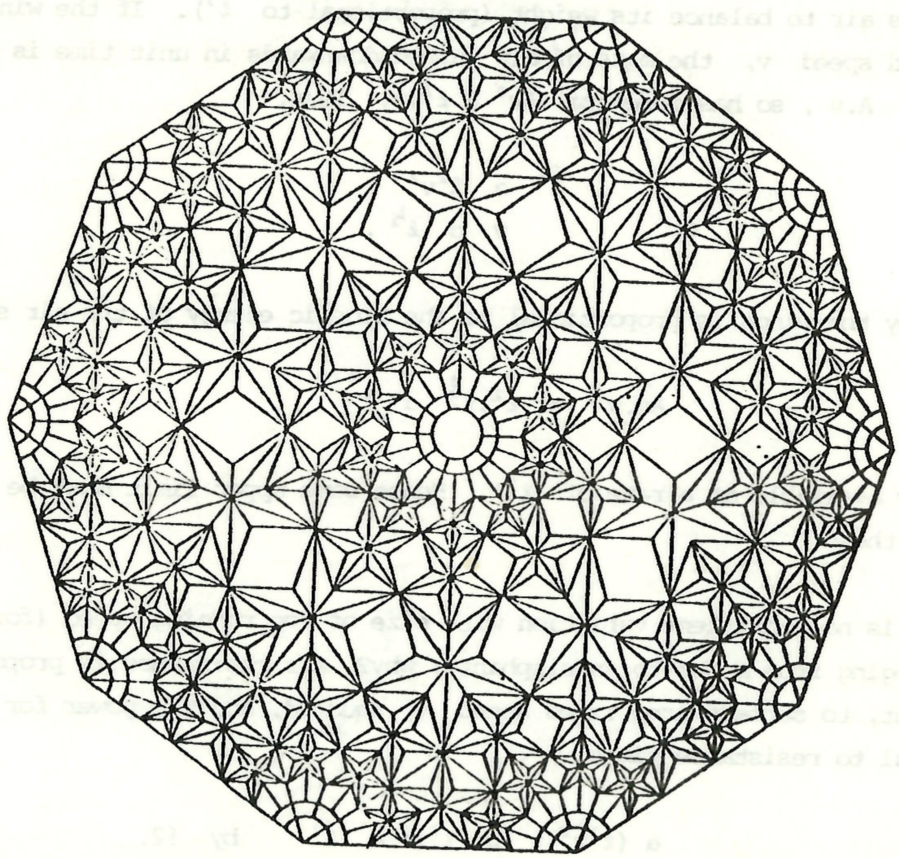
$$\propto (\ell^2 v^2)v \propto \ell^2 v^3 \propto \ell^2 \qquad \text{by §2.}$$

Hence v^3 is independent of size (as we have seen before).

§4. Really? Can you decide how stamina depends on ℓ? Does this refine any of the previous arguments? Do you believe, in the calculation of power, that v^2 is independent of ℓ? Can you convince yourself that the suppression of calculus in these formulae does not matter? Can you think of a situation in which the first order approximations above become inaccurate?



THE END



There are one hundred pentagonal Stars in this pattern. We may continue their construction towards the centre and towards the vertices of the ten sided polygon. What portion of the large polygon will be covered if we cotninue our construction to "infinity" ?