SOLUTIONS TO PROBLEMS FROM VOLUME 17, NUMBER 1.

Q. 479. If a679b is a five-digit number (in base 10) which is divisible by 72, determine a and b.

SOLUTION: Andrew Jenkins (North Sydney Boys' High School), writes:-

As the number is divisible by 72, it must be divisible by both 8 and 9, which have no common factors. For a number to be divisible by 8, the last three digits must be divisible by 8, therefore 79b must be divisible by 8, making b equal to 2.

For a number to be divisible by 9, its digits must add up to a multiple of 9. The digital sum of 6792 is 24, therefore a is equal to 3, making the number 36792.

Correct Solutions also by A. Cherry (Finley High School), J. Taylor (Hornsby Girls' High School), and S.S. Wadhwa (Ashfield Boys' High School).

Q. 480. The numbers from 1 to 50 are printed on cards. The cards are shuffled and then laid out face up in 5 rows of 10 cards each. The cards in each row are rearranged to make them increase from left to right. The cards in each column are then rearranged to make them increase from top to bottom. In the final arrangement, do the cards in the rows still increase from left to right?

SOLUTION: The answer is yes, as the following argument shows. In the contrary case there would be some row (say the "i"th) containing, after the second ordering, two neighbouring cards (say in the jth and (j + 1)th columns) with the number on the card in column j greater then that on the card in column j + 1. That is the "ith" smallest card in column j, is larger than the "i"th smallest card in column j + 1. We shall prove that this is impossible after the first ordering, and hence also after the second. Let the "i"th smallest card in column j + 1 carry the number a. Then there are i - 1 smaller numbers in column j + 1, and the neighbouring cards in column j are even smaller. Hence there are at least i numbers in column j which are smaller than a.

An incomplete argument was received from J. Taylor (Hornsby Girls' High School).

Q. 481. Among all triangles having (i) a fixed angle A and (ii) an inscribed circle of fixed radius r, determine which triangle has the least perimeter.

SOLUTION: In Figure 1, the perimeter of ΔABC is equal to $AF + AE + 2BD + 2DC$, since $BF = BD$ and $CD = CE$. Since AF and AE are fixed, we have to determine the position of BC to minimise $BD + DC = r(\cot\frac{B}{2} + \cot\frac{C}{2})$.

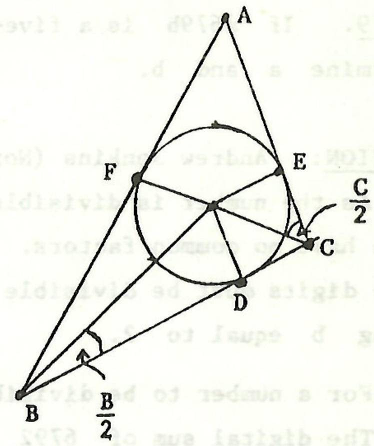


Figure 1

$$\text{Now } \cot\frac{B}{2} + \cot\frac{C}{2} = \frac{\sin\frac{C}{2}\cos\frac{B}{2} + \cos\frac{C}{2}\sin\frac{B}{2}}{\sin\frac{C}{2}\sin\frac{B}{2}}$$

$$= \frac{2\sin(\frac{B}{2} + \frac{C}{2})}{\cos(\frac{C}{2} - \frac{B}{2}) - \cos(\frac{C}{2} + \frac{B}{2})} = \frac{2\cos\frac{A}{2}}{\cos(\frac{C}{2} - \frac{B}{2}) - \sin\frac{A}{2}}$$

Since A is fixed, this achieves its least value when the denominator is greatest i.e. when $\cos(\frac{C}{2} - \frac{B}{2}) = 1$, forcing $\frac{C}{2} - \frac{B}{2} = 0$. We have proved that the minimum perimeter occurs when $B = C$, i.e. when the triangle is isosceles.

Correct Solution from S.S. Wadhwa (Ashfield Boys' High School).

Q. 482. A gambling student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring exactly n points is $\frac{1}{3}\{2 + (-\frac{1}{2})^n\}$.

SOLUTION: S.S. Wadhwa (Ashfield Boys' High School) writes:-

Proof by induction. Assume $n = 1$: Only possible way is to throw H $\therefore$ Pr $= 1/2$.

By formula: $1/3\{2 + (-1/2)^1\} = 1/3.3/2 = 1/2$.

Assume $n = 2$: Two possible ways: H followed by H $\therefore$ Pr $= (1/2)^2 = 1/4$ or T $\therefore$ Pr $= 1/2$ $\therefore$ Required Pr $= 1/4 + 1/2 = 3/4$,

By formula: $1/3\{2 + (-1/2)^2\} = 1/3.9/4 = 3/4$.

Assume that formula works as far as getting exactly k points. To get exactly (k + 1) points, we must throw H if we have exactly k points or we must throw T if we have exactly (k - 1) points.

$$\therefore \text{ Pr} = \underbrace{1/3\{2 + (-1/2)^k\}.1/2}_{\text{Pr of H}} + \underbrace{1/2\{2 + (-1/2)^{k-1}\}.1/2}_{\text{Pr of T}}$$

(continued over)

$$= 1/6\{2 + (-1/2)^k + 2 + (-1/2)^{k-1}\}$$

$$\therefore \quad Pr = 1/6\{4 + (-1/2)^{k-1}(-1/2 + 1)\}$$

$$\therefore \quad Pr = 1/6\{4 + (-1/2)^{k-1}.(1/2)\}$$

$$= 1/6\{4 + (-1/2)^{k+1}.2\}$$

$$\therefore \quad Pr = 1/3\{2 + (-1/2)^{k+1}\}$$

i.e., the formula works for exactly $(k + 1)$ points and since it works for $k = 1,2$ therefore it must work for $k = 3,4,5, \ldots,$ and so on, hence it must work for $k = n$.

Q. 483. On Mathland T.V., a political commentator's summary of an election result was as follows:

"A Labour majority of 1729 last time had been turned into a Conservative majority of 1654 in this election, and the Conservative candidate has obtained 38% of the poll. Labour has taken second place. The Liberal has obtained only 14% of the poll and has been beaten into bottom place by the M.N.P. candidate, who has 50 more votes then the Liberal."

Given that there were just four candidates and that all the figures quoted were exact, determine the number of votes polled for each cnadidate.

SOLUTION: A. Cherry (Finley High School) writes:-

Let 'x' be the total number of voters.

$\therefore$ the number of Conservative voters is $\frac{38}{100}.x$

the number of Labour voters is $(\frac{38}{100}.x - 1654)$

the number of M.N.P. voters is $(\frac{14}{100}.x + 50)$

The number of Liberal voters is $\frac{14}{100}.x$

$$\therefore \quad x = \frac{38}{100}.x + (\frac{38}{100}.x - 1654) + (\frac{14}{100}.x + 50) + \frac{14}{100}.x$$

$$\therefore \quad 100x = 104x - 160400$$

$$\therefore \quad 4x = 160400$$

$$\therefore \quad x = 40100$$

The number of voters for Conservatives, Labour, M.N.P. and Liberal are, respectively, 15238, 13584, 5664 and 5614.

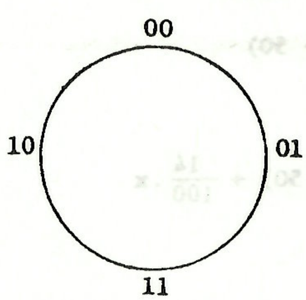
Correct Solutions also from S.S. Wadhwa (Ashfield Boys' High School), P. Curran (St. Patricks' College, Goulburn) and J. Taylor (Hornsby Girls' High School).

Q. 484. This question appeared as follows:

Alan and Bill are serving at a hot drinks stall which provides only tea and coffee. The price of a cup of coffee is greater than that of a cup of tea, both prices being a whole number of pence. As each customer is served the amount paid is entered on a list. Alan sees that Bill has entered amounts of 30p and 61p on the list and knows that these are both incorrect as no transactions could give these totals. Alan himself has just served a customer with more than one drink and has, correctly, charged 19p. How much is a cup of coffee?

SOLUTION: Alas, I owe infuriated readers a humble apology for not having noticed a serious typographical error in this question, even after Mr. G. Davis (North Sydney Technical College) had questioned whether a solution existed. The amount of 30p should have read 39p. In a later communication Mr. Davis supplied the following argument to prove that no solution was possible:- The price of a cup of tea or coffee cannot be a factor of 30, nor of $30 - 19 = 11$, nor $61 - 19 = 42$. This eliminates 1, 2, 3, 5, 6, 7, 10, 11, 14 and 15. Similarly $4 = 61 - 3 \times 19$ must be eliminated. It is then impossible to find a transaction amounting to 19p from the remaining possible prices 8, 9, 12, 13, 16, 17, 18. Applying a similar argument to the corrected question, you will have no difficulty in eliminating factors of 39, $39 - 19$, and $61 - 19$, viz. 1, 2, 3, 4, 5, 6, 7, 10, 13, 14. The only way of expressing 19 as a sum of prices not in that list is $19 = 11 + 8$, and it is not difficult to verify that no transactions with these prices can give 39p or 61p.

Q. 485. There are four 2-digit binary numbers, namely 00, 01, 10 and 11. In the diagram they are placed round the circumference of a circle in such a way that any two adjacent numbers differ in only one digit.



Determine whether the eight 3-digit binary numbers can be placed round the circumference of a circle in such a way that any two adjacent numbers differ in only one digit. Is a similar argument possible for the sixteen 4-digit binary numbers?

SOLUTION: The following orderings work for 3-digit and 4-digit binary numbers

000, 001, 011, 010, 110, 111, 101, 100, (000)

0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000, (0000).

Note that the first half of the 4-digit list is obtained by adjoining a preceding 0 to the entries in the 3-digit list. The second half is constructed by adjoining a preceding 1 instead, and also reversing the order. The same construction can be used to obtain a solution for 5-digit numbers from the 4-digit list, and so on.

Correct Solutions from A. Cherry (Finley High School), J. Taylor (Hornsby Girls' High School), and S.S. Wadhwa (Ashfield Boys' High School).

Q. 486. A photograph shows four men, all the same height, standing in a straight line. The image of the first man is 3 cm high and the image of the fourth man is 2 cm high. One of the other two men was standing the same distance from the first man as from the fourth man: how high is his image? The remaining man's image is 2.5 cm high and he was 3 m from the first man. How far was he standing from the fourth man? Explain your working. (You may assume that all the images of the men are parallel to one another.)

SOLUTION:

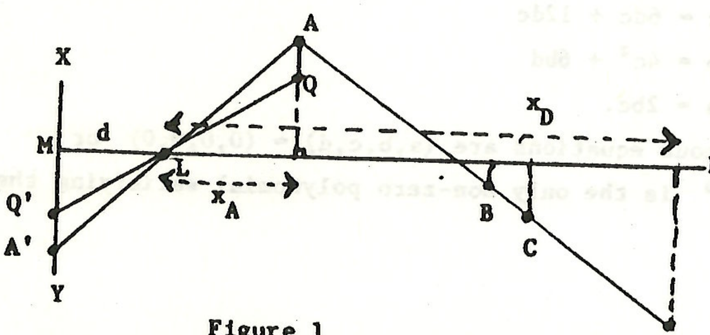


Figure 1

In the figure, XY represents the plane of the film, and L the centre of the lens of the camera, MLN being a line at rightangles to XY. If d cm denotes the distance LM, and x_A the distance from L to a plane AQ at rightangles to LN, one easily sees by similar triangles that the ratio of the lengths of any line segment AQ in that plane to the length of its image A'Q' satisfies $\frac{AQ}{A'Q'} = \frac{x_A}{d}$. Now let A,B,C,D denote the positions of the four men, in planes perpendicular to LN, the axis of the camera at distances x_A, x_B, x_C and x_D (resply) from the lens. (N.B. Since the images of the men are parallel to one another, the axis of the camera must be horizontal). Take AQ = h metres, the height of

continued over

each man, and A'Q' = 3 cms, obtaining $3x_A = dh$. Similarly $2x_D = dh$., whence $x_D = \frac{3}{2}x_A$. If C is standing midway between A and D, it is clear that $x_C = \frac{x_A + x_D}{2}$. Thus $x_C = \frac{5}{4}x_A$, and the height of C's image is given by $\frac{dh}{x_C} = 4 \times \frac{3x_A}{5x_A} = \frac{12}{5}$ cms.

Finally, the last man B has an image of height 2.5 cm $\therefore$ $x_B = \frac{dh}{2.5} = \frac{3x_A}{2.5} = 1.2x_A$

Thus $x_B - x_A : x_D - x_B = .2x_A : .3x_A = 2:3$ and by similar triangles again, AB/BD = 2/3. Since AB is 3m this gives BD = 4½ metres.

Correct Solution from A. Cherry (Finley High School).

Q. 487. Find all polynomials P(x) satisfying the relation

$$P(2x) = P'(x) \cdot P''(x)$$

SOLUTION: Obviously the zero polynomial is one solution. If $P(x) \neq 0$ let its degree be n. Them P'(x) is of degree (n - 1), P"(x) is of degree n - 2; and P(2x) has degree n. Therefore n = (n - 1) + (n - 2) giving n = 3. So we can assume an expression of the form $a + bx + cx^2 + dx^3$ for P(x). Substitution gives

$$a + 2bx + 4cx^2 + 8dx^3 = (b + 2cx + 3dx^2)(2c + 6dx).$$

Equating coefficients we obtain:-

$$8d = 18d^2$$
$$4c = 6dc + 12dc$$
$$2b = 4c^2 + 6bd$$
$$a = 2bc.$$

The only solutions of these simultaneous equations are (a,b,c,d) = (0,0,0,0) or $(a,b,c,d) = (0,0,0,\frac{4}{9})$. Hence $\frac{4}{9}x^3$ is the only non-zero polynomial satisfying the given relation.

Q. 488. A and B lie on one side of a square of side length 1, and C lies on the opposite side Let R be the radius of the circumcircle of ΔABC, and let $h < R < k$. What values of h and k makes the relation true for all positions of the triangle?

SOLUTION: The relation $h < R < k$ is true for all positions of the triangle when h is any real number $\leq \frac{1}{2}$ and k is any real number ≥ 1. In Figure 1, O is the circumcentre of the triangle ABC, $OA + OC \geq AC \geq 1$ i.e. $2R \geq 1$. Also $OB + OC \geq BC \geq 1$. Equality is obtained in either case if O is the midpoint of AC or BC respectively. Since O cannot simultaneously be the midpoint of each of the two sides AC and BC of the triangle, one of the inequalities can be replaced by strict inequality. Thus $R > \frac{1}{2}$ for every position of ΔABC.

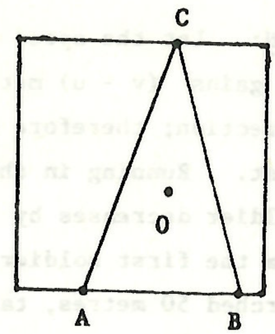


Figure 1

Taking the triangle $A_1B_1C_1$, in Figure 2, the hypotenuse B_1C_1 is the diameter of the circumcircle, so that the value of R, O_1B_1, can be made as close to $\frac{1}{2}$ as desired by moving B_1 sufficiently close to A_1. We have now shown that the largest allowable value for h is $\frac{1}{2}$.

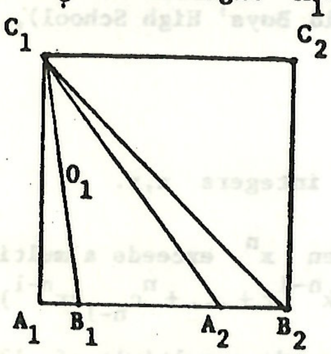


Figure 2

Using the formula

$$R = \frac{b}{2 \sin B} = \frac{AC}{2 \sin A\hat{B}C}$$

from trigonometry, and noting that the largest possible value of AC is $\sqrt{2}$ (the length of the diagonal of the square, achieved with A at A_1, C at C_2), and the smallest value of sin ABC is $\sin 45° = \frac{1}{\sqrt{2}}$ (achieved with C at C_1, B at B_2), we see that $R \leq \frac{\sqrt{2}}{2.\sqrt{2}} = 1$.

Since the maximum value of AC and the minimum value of $\sin A\hat{B}C$ are not simultaneously achieved one of the inequalities must be strict, and we can sharpen the result to $R < 1$. The triangle, $\Delta A_2B_2C_1$, has $\sin B_2 = \frac{1}{\sqrt{2}}$, and by letting A_2 approach B_2 we can make A_2C_1 as near to $\sqrt{2}$ as desired. Hence R can be made as close to 1 as desired. Thus the smallest value for k is 1.

Q. 489. A squad of soldiers are marching in Indian file at constant speed. The length of the file is 50 metres. A dog runs at constant speed from the last soldier to the first, then immediately turns round and returns to the last soldier. During this time the soldiers have moved forward 50 metres. How far has the dog run?

SOLUTION: Let the speed of the dog be v m/sec, that of the soldiers u m/sec. The dog gains (v - u) metres on the soldiers each second when running in the same direction; therefore he takes $\frac{50}{v-u}$ seconds to run from the last soldier to the first. Running in the opposite direction the distance between the dog and the last soldier decreases by (v + u) metres each second; he takes $\frac{50}{v+u}$ seconds to run from the first soldier back to the last. While doing both trips the soldiers have marched 50 metres, taking 50/u seconds. Hence $\frac{50}{v-u} + \frac{50}{v+u} = \frac{50}{u}$;
yielding $\frac{1}{v/u - 1} + \frac{1}{v/u + 1} = 1$, and $(\frac{v}{u})^2 - 2(\frac{v}{u}) - 1 = 0$ after simplification.

Solving; $\frac{v}{u} = 1 + \sqrt{2}$ (discarding the meaningless negative solution). Hence while the soldiers walk 50 metres the dog runs $50(\frac{v}{u})$metres $= 50(1 + \sqrt{2})$ metres.

Beautiful solutions received from - S.S. Wadhwa (Ashfield Boys' High School), and A. Jenkins (North Sydney Boys' High School).

Q. 490. Show that $x^3 + y^4 = 19^{19}$ has no solution in integers x,y.

SOLUTION: If x exceeds a multiple of 13 by r then x^n exceeds a multiple of 13 by r^n. [Proof. $(13k + r)^n = 13(13^{n-1}k^n + {}^nc_1\, 13^{n-2}k^{n-1}r + \ldots + {}^nc_{n-1}kr^{n-1}) + r^n$] Using this it is not difficult to verify that if x exceeds a multiple of 13 by 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, or 12 then x^3 exceeds at multiple of 13 by 0, 1, 8, 1, 12, 8, 8, 5, 5, 1, 12, 5, or 12 respectively, and x^4 exceeds a multiple of 13 by 0, 1, 3, 3, 9, 1, 9, 9, 1, 9, 3, 3, or 1 respectively. It follows that it is impossible for $x^3 + y^4$ to exceed a multiple of 13 by 7 for any integers x and y. (Check that neither 7 nor 20 is expressible as the sum of two numbers, one in each of the last two lists.)

However (you've guessed it) it turns out that 19^{19} exceeds a multiple of 13 by 7. By the first remark above, the remainder when 19^n is divided by 13 is the same as the remainder when 6^n is divided by 13. For n = 2, 3, 4, 8, 16 & 19 this is seen to be 10, 8, 9, 10, 9, and 7 respectively.

Solvers of earlier problems.

The following correct problem solutions were not acknowledged in the last issue of Parabola:-

N. Brown (Dickson College) submitted correct solutions to problems 467, 468, 469, 470, 473, 474, 475, 476, 478, and 480 and S.S. Wadhwa (Ashfield Boys; High School) to 467, 468, 469, and 480.