THE COASTLINE OF AUSTRALIA

BY

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For hundreds of years, it was accepted in international law that the territorial waters of any country extended three miles from the shore. That particular distance was probably chosen as being the extreme range of the sort of cannon that were used in fortifications. More recently, there has been a tendency for countries to claim jurisdiction up to a two hundred mile limit. When this question came up a few years ago in Australia, the Royal Australian Navy was concerned because the change would mean that the Fisheries patrols would have to cover a much larger area of sea. How much is the extra area? People said it was 197 (= 200 - 3) miles times the length of the coastline. They were wrong because, as all Parabola readers know, the length of the coastline is infinity. However, there is some truth in the idea.

For a start, consider a simpler situation. Let E be the blob in figure 1. (More precisely, suppose E is convex, that is a line segment joining any two points of E lies entirely in E.) Let $E(r)$ be the set obtained by adding all points at distance r or less from E. Let us denote the area of the blob E by A and its

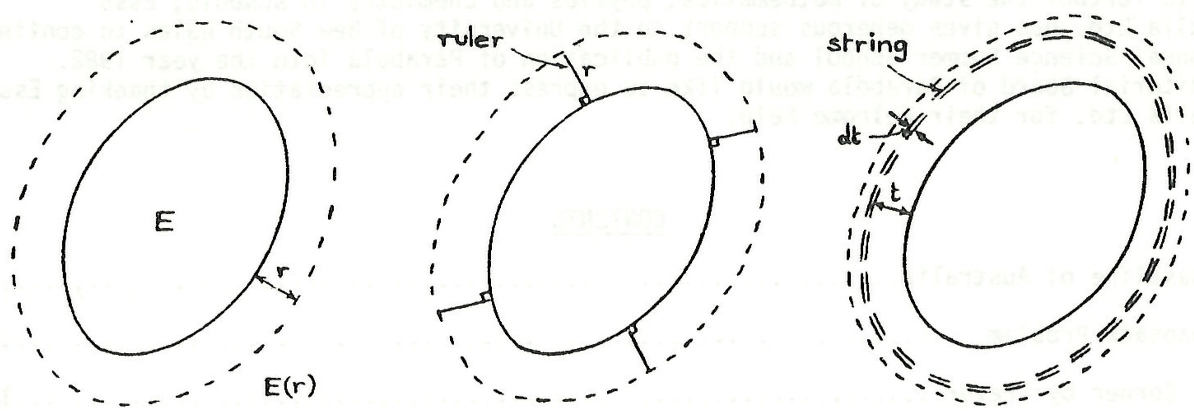

Figure 1.

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perimeter by S. What can we say about the area and perimeter of $E(r)$? To measure the perimeter of $E(r)$, imagine a ruler of length r moving around E and always perpendicular to the boundary. As the ruler moves around E, the outer end describes the perimeter of $E(r)$. To see how far it moves, we can break the motion into two components. Firstly, the outer end must follow the inner end which moves a distance S round the perimeter of E. Secondly, the outer end makes one complete revolution about the inner end and the contribution from this component is just the circumference of a circle of radius r, namely $2\pi r$. So the perimeter of $E(r)$ is $S(r) = S + 2\pi r$. To measure the area of $E(r)$, imagine E surrounded by lots of loops of string. The loop shown in the figure has length $S(t)$ and thickness dt and so covers an area $S(t)dt$. To get the area of $E(r)$, we have to add the areas of all these loops, as t runs from 0 to r, to the area of E. For very thin string, the sum becomes an integral, so the area of $E(r)$ is

$$A(r) = A + \int_0^r S(t)dt = A + \int_0^r (S + 2\pi t)dt = A + Sr + \pi r^2.$$

However, if the set E has bumps, that is E is not convex, then things are more difficult. For example, the perimeter of E may be infinite. We shall see that the expanded sets $E(r)$ are always smoother than E. For example, if E is bounded, that is it does not extend indefinitely in any direction, then $E(r)$ has a finite perimeter for any positive value of r. To prove this, imagine yourself sailing anticlockwise round Australia on the three mile limit, keeping a reasonable speed of at least one knot. You would probably get round in a few years. How do we know for

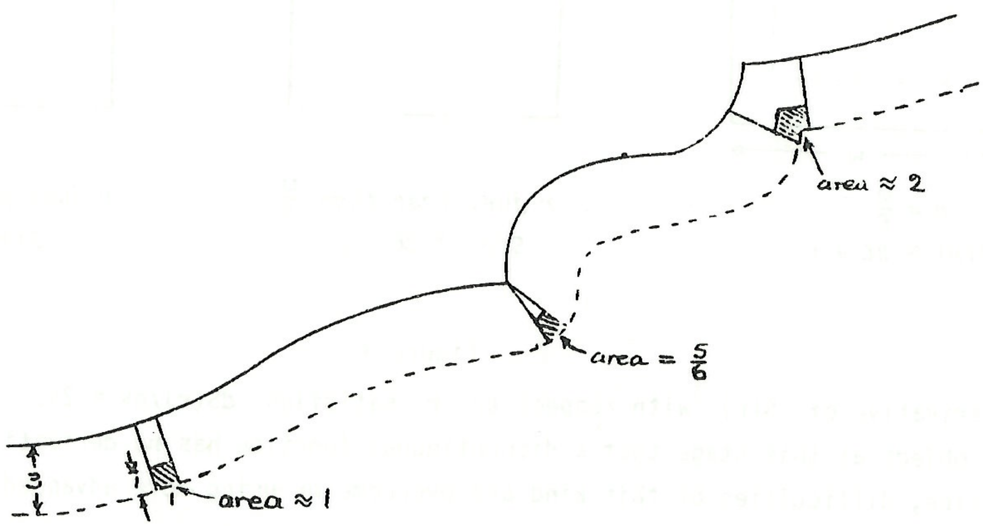

Figure 2.

certain that you would take only a finite time? Imagine you are making a chart of all the waters up to one mile on the port (inshore) side of your track. In each hour, you will travel at least one mile and chart at least 5/6 of a square mile. (See figure 2.) As you never chart the same bit of sea twice and as the total area of the territorial waters is finite, you must finish in a finite time. That is the general idea of the proof, but of course there are some details to be tidied up. To do the job properly, you will have to detour to make the circuit of Tasmania and various other islands and you will have to go clockwise round Port Philip Bay. As you can see in figure 2, the boundary of the three mile limit may have sharp turns to starboard when you go round anticlockwise, but the turns to port are only very gentle; in fact, their radius of curvature is at least three miles.

Suppose the set E is connected, that is it is not in two separate bits. The extended set E(r) has a finite perimeter S(r) for any positive value of r. Now S(r) as a function of r may decrease quickly and it can even decrease discontinuously as shown in figure 3, but it cannot increase too fast. It is not too hard to see that

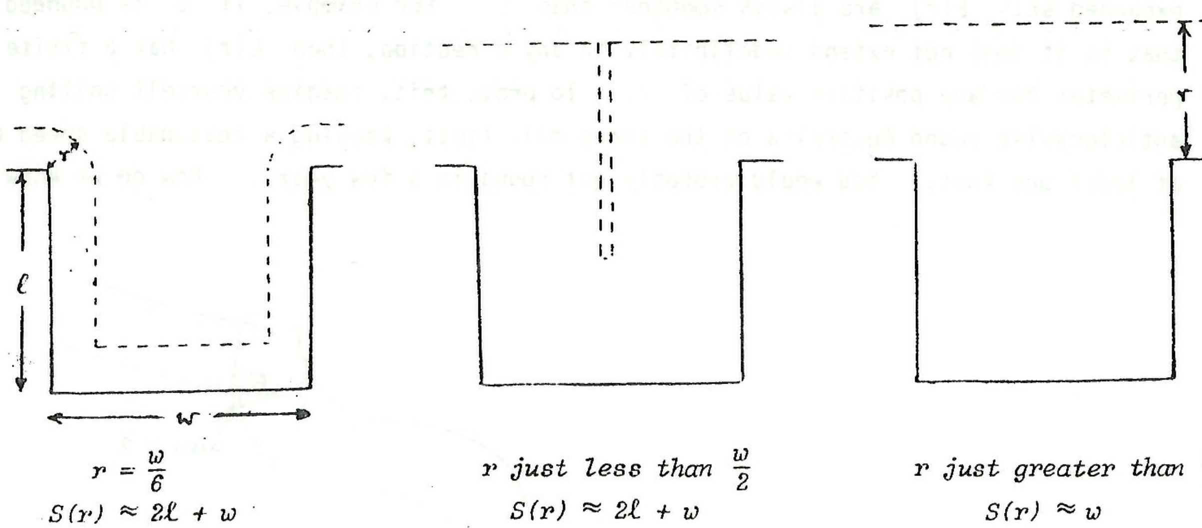

Figure 3.

the derivative of S(r) with respect to r satisfies $dS(r)/dr \leqslant 2\pi$. Some people might object at this stage that a discontinuous function has no derivative, but take no notice; difficulties of this kind are overcome by using more advanced mathematics.

Now to return to our original question about how the area A(r) changes with r. First, we see that the derivative of A(r) with respect to r is just S(r). You can see this by adding a loop of thin string around the boundary of $E(r)$ and calculating the change in area. So we have

$$\frac{dS(r)}{dr} \leqslant 2\pi \quad \text{and} \quad \frac{dA(r)}{dr} = S(r).$$

Let us denote the area and perimeter when $r = r_0$ by $A(r_0) = A_0$ and $S(r_0) = S_0$ respectively. If we integrate the inequality above from r_0 to r, we obtain

$$S(r) - S_0 \leqslant 2\pi(r - r_0), \quad \text{or} \quad \frac{dA(r)}{dr} \leqslant S_0 + 2\pi(r - r_0).$$

Integrating again gives

$$A(r) - A_0 \leqslant S_0(r - r_0) + \pi(r - r_0)^2.$$

We can even get an inequality in the other direction. Suppose that E has maximum diameter d, so that E contains two points P and Q this distance apart, but no pair of points any further apart. (See figure 4.). Then $E(r_0)$ had diameter $d + 2r_0$. Let P_0 and Q_0 be two points of $E(r_0)$ which are a distance $d + 2r_0$

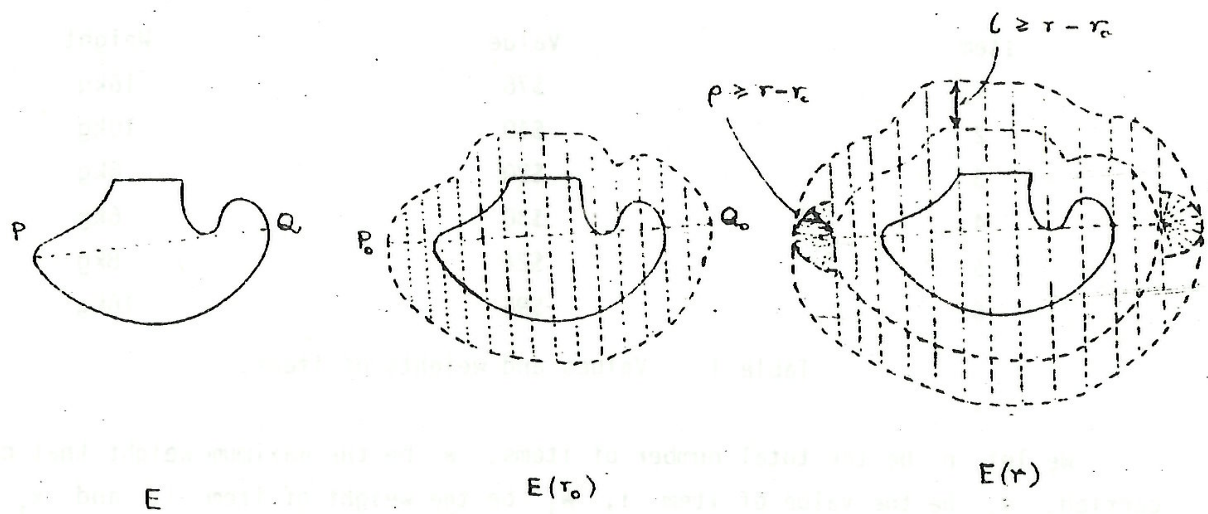

Figure 4.

apart and cut the area into strips by lines perpendicular to P_0Q_0. By comparing areas, you will see that

$$A(r) - A_0 \geqslant 2(d + 2r_0)r - r_0) + \pi(r - r_0)^2 .$$

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