H.S.C. CORNER (BY TREVOR)

Trevor's Corner is one of the most popular and useful features of our magazine. Starting in this issue we will number the problems indicating the year as well. We would appreciate receiving your comments, queries and problems.

The new format of the 4-unit paper contains provisions for a number of harder questions on the 3-unit syllabus, as well as questions on the 4-unit material. Question 7 is a good example of rather hard questions on the 3-unit course.

82.1. (i) Find all x such that $\sin x = \cos 5x$, and $0 < x < \pi$.

This trig equation becomes quite easy when we realise that $\cos 5x = \sin(\frac{\pi}{2} - 5x)$. Thus

$$\sin x = \sin(\tfrac{\pi}{2} - 5x). \qquad (1)$$

Obviously, one set of solutions is given by $x = \frac{\pi}{2} - 5x + 2n\pi$, for $n = 0, 1, 2, \ldots$ etc., so that $6x = \frac{\pi}{2} + 2n\pi$, or $x = \frac{\pi}{2} + n\frac{\pi}{3}$. The three solutions in the required range are, therefore; $\boxed{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{12}.}$ However, note that $\sin\theta = \sin(\pi - \theta)$, and another set of solutions is given by

$$\pi - x = \frac{\pi}{2} - 5x + 2n\pi, \quad n = 0, 1, 2, \ldots$$

leading to the two solutions (for $n = 1, 2$) $\boxed{\frac{3\pi}{8}, \frac{7\pi}{8}.}$ This gives a total of five solutions of (i) in the range $0 < x < \pi$.

An ingenious alternative method is to write (i) as:

$$\sin(3x - 2x) = \cos(3x + 2x)$$

i.e. $$\sin 3x \cos 2x - \cos 3x \sin 2x = \cos 3x \cos 2x - \sin 3x \sin 2x.$$

Dividing by $\cos 2x \cos 3x$ yields

$$(\tan 3x - 1)(\tan 2x + 1) = 0$$

whence either $3x = \frac{\pi}{4} + n$, or $2x = -\frac{\pi}{4} + n$, and the results follow.

82.2 (ii) Nine persons gather to play football by forming two teams of four to play each other, the remaining person to act as referee. In how many different ways can the teams be formed?

If two particular persons are not to be in the same team, how many ways are there then to choose the teams?

The simplest way is first to choose the ref (9 ways). For each ref., there are ${}^8C_4$ ways to choose team A, and then there remain 4 persons for team B. There are $9 \times {}^8C_4$ such choices. But the same group of 4 persons could be chosen for either team A or team B. Thus the number of different ways of forming the teams is $\frac{1}{2} \times 9 \times {}^8C_4 = \boxed{315}$.

The second part of the question is best attempted by first finding the number of ways these two players must be in the same team. Suppose the two players are in team A, then after the ref. is chosen, there are ${}^6C_2$ ways of forming team A (since two players must be chosen to complete the team). Thus the number of ways the two players are not on the same team is $315 - 7 \times {}^6C_2 = \boxed{210}$ ways.

82.3 (iii) In this question we have to prove that when x,y,z are real and not all equal,

$$x^2 + y^2 + z^2 > yz + zx + xy;$$

and deduce that when also $x + y + z = 1$, then $yz + zx + xy < \frac{1}{3}$.

Everyone knows that if $a \neq b$ and a,b are real, then $(a - b)^2 > 0$, yielding the famous inequality $a^2 + b^2 > 2ab$. Hence $(x^2 + y^2) + (y^2 + z^2) + (z^2 + x^2) > 2xy + 2yz + 2zx$, and the required result follows. Moreover, when $x + y + z = 1$

$$1 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) > 3(xy + yz + zx).$$

The three unit paper also contained some interesting material, including

82.4 [3 unit Q. 5 (ii)]: The polynomial $(x - a)^3 + b$ is zero at $x = 1$, and when divided by x, the remainder is $- 7$. Find all possible values of the pair (a,b).

Let $f(x) = (x - a)^3 + b$, and use the remainder theorem twice

$$f(1) = (1 - a)^3 + b = 0, \quad \text{and} \quad f(0) = - a^3 + b = - 7.$$

Thus $(1 - a)^3 + a^3 = 7$, whence $a^2 - a - 2 = 0$, so that the pairs of values of (a,b) are $(2,1)$ and $(-1, -8)$.

82.5 [3 unit, Q. 8 (ii)]. Two players X,Y of about equal ability play each other twice in a chess competition, each having the white pieces once. In the competition a player who wins scores 1 point, a loser scores 0 points, and a drawn game yields $\frac{1}{2}$ point to each player. Given that the probability that White wins is 0.5, that Black wins is 0.3, what is the probability that from the two games

a) X scores two points,

b) X scores less that $1\frac{1}{2}$ points.

Obviously the probability of a draw is $1 - .5 - .3 = 0.2$.

To score 2 points X must win both games (once as White, and once as Black) - with probability $0.5 \times 0.3 = \boxed{0.15}$.

To score $1\frac{1}{2}$ points X must win one and draw one. When X wins as White the probability is $0.5 \times 0.2 = 0.1$. When X wins as Black the probability is $0.3 \times 0.2 = 0.06$.

Hence the probability of scoring less than $1\frac{1}{2}$ points is $1 - 0.31 = \boxed{0.69}$.

Two examples you might wish to try are:

82.6. Find all values of x in the range $0 < x < \pi$ such that

$$2 \cos^2 2x + \cos 6x = 1.$$

82.7. a) Ten persons are to play doubles tennis, forming two sets of four, and two umpires. How many different arrangements of the players as there?

b) If there are 5 men and 5 women, how many mixed doubles arrangements are there? [i.e. each pair consists of 1 man and 1 woman.]