PROBLEMS

Q. 515. I have two different integers > 1. I inform Sam and Pam of this fact and I tell Sam the sum of my two numbers and I tell Pam their product. The following dialogue then occurs:

Pam: I can't determine the numbers.

Sam: The sum is less than 23.

Pam: Now I know the numbers.

Sam: Now I know the numbers too.

What are the numbers?

Q. 516. Solve the following equation:

$$x = 1 + \cfrac{2}{1 + \cfrac{2}{1 + \cdots \cfrac{\vdots}{1 + \cfrac{2}{1 + \cfrac{2}{x}}}}}$$

(There are n vinculi (division lines) in the expression on the RHS).

Q. 517. Let P,Q,R,S be four points on the sides of a triangle ABC such that $PQRS$ is a rectangle. Show that the maximum area of the rectangle $PQRS$ is half the area of the triangle ABC.

Q. 518. Find a polynomial with integer coefficients having $\sqrt{2} + \sqrt{3}$ and $\sqrt{2} + \sqrt[3]{3}$ as roots.

Q. 519. Let S be a set of real numbers containing at least one non-zero number, and possessing the following property:

For every pair (r,s) of numbers in S (not necessarily different), $r - s$ is in S, and, if $s = 0$, the quotient r/s is in S.

Prove that every rational number is in S.

Q. 520. Let DEF be the pedal triangle of the acute angled trianlge ABC (i.e., D,E,F are the feet of the perpendiculars from A,B,C to BC, CA, AB respectively). Show that the greatest angle of triangle DEF is at least as large as the greatest angle of ABC. When does one have equality?

Q. 521. Find all pairs of integers x,y such that

$$x^3 - y^3 = 1729$$

Prove that there are no others.

Q. 522. Show that there exist two constants A and B $(A > B)$ such that, if f is the function defined for all $x \neq -1/B$ by

$$f(x) = (1 + Ax)/(1 + Bx),$$

then the expression $f(1/(1 + 2x))/f(x)$ is independent of x (i.e., constant wherever it is defined). Next, consider the list $(a(n))$ defined by

$$a(0) = 1, \quad a(n + 1) = 1/(1 + 2a(n)) \quad \text{for } n = 0,1,2, \ldots$$

By considering also the list $f(a(0))$, $f(a(1))$, $f(a(2))$, ... find a formula giving the value of $a(n)$ for every n.

Q. 523. Let S be the set of polynomials of the form

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

such that $|f(x)| \leqslant 1$ whenever $|x| \leqslant 1$. Show that there exists a number M such that $|a_3| \leqslant M$ for all polynomials in S. Try to find a value for M as small as possible.

Q. 524. Show that, for every positive integer n,

$$2(\sqrt{(n + 1)} - \sqrt{n}) < 1/\sqrt{n} < 2(\sqrt{n} - \sqrt{(n - 1)}).$$

Find the largest integer less than

$$1 + 1/\sqrt{2} + 1/\sqrt{3} + \ldots + 1/\sqrt{10000}\ .$$

Q. 525. A point in the Cartesian plane is called an integer point if its coordinates (x,y), are both integers. Denote by $A(R)$ the number of integer points lying inside the disc of radius R, centre the origin.

a) Show that the expression $A(R)/R^2$ approaches a limiting value, ℓ, then R tends to infinity. Find ℓ.

b) For all R, let

$$B(R) = A(R) - \ell R^2$$

Then for $k = 2$, it is certainly true that $B(R)/R^k$ approaches 0 as R tends to infinity. For which values of k less than 2 is this statement still true?

Q. 526. A box is locked with several padlocks, all of which must be opened to open the box, and all of which have different keys. Five people each have keys to some of the locks. No two of the five can open the box but any three of them can. What is the smallest number of locks for which this is possible?

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