FINITE LISTS AND THE PROPOSITIONAL CALCULUS*

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Part 1

I was spending the weekend at Woodful Towers when wealthy old Sir Joshua Woodful was horribly murdered in the library. You probably read about it - it was in all the papers at the time. Immediately I called in my old friend Atlas Pierrot who was passing through the nearby village at the time on a walking holiday.

So it was that we were all gathered in the library with Inspector Hoey on a stormy night. The wind howled as we sat gazing at the large stain on the carpet. Atlas spoke, "I have made a complete list of the suspects". He placed a piece of paper on the table, on which was written, 'The Squire, The Vicar, The Cook, The Butler, The Maid, The Gardener'.

"Now for the alibis", said the inspector. "The vicar was taking evensong at the time of the murder".

Atlas crossed the name off his list.

"The cook was at the Women's Institute".

Atlas crossed off the name.

"The squire and the gardener were playing darts at the Pig and Whistle".

Two more names were crossed off.

"The maid was at the Roxy cinema at Woodfulton", I added.

A smile flickered across Atlas's face as he crossed off another name.

"Well, inspector, when we have eliminated the impossible whatever remains, however improbable, must be the truth'

"You sould like Mr. Holmes", remarked the inspector.

"So the Butler did it", I gasped in astonishment. Just then the lights went out.

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Afterwards I talked to Atlas about this method of making a list of all the possibilities and working through them.

"There is the restriction on its use in that there must only be a finite number of possibilities", he explained. "However, many situations do meet this condition, in particular the mathematical model of the logic of propositions known as the propositional calculus".

"Does this model exactly mirror the real situation?" I asked.

"No. To do this requires a complex model and so we compromise and select the most realistic simple model.

We first have simple propositions or sentences which are statements of fact and are either true or false; e.g.

'snow is white' is true,

'a cube has 7 faces' is false.

Next we select the logical words we want to have in our model. In this model we have

not, and, or, if-then.

We use these to build up complex sentences; e.g.,

'if snow is white then a cube has 7 faces', 'not snow is white',"

"Just a minute, shouldn't that be 'snow is not white'?"

"Yes, but the model would be much more complex if it had to consider the correct grammatical position of the 'not' so we always put it at the front. Similarly we depart from the common usage by using brackets in the model, e.g.,

'if snow is white then (a cube has 7 faces or London is in England)'.

This is to avoid the ambiguity which would result if the brackets were dropped".

"Dash it all, I would never use a sentence like that".

"Correct, but it's the same old problem. If the model only included sentences we commonly use, then it would have to be very complex, so we simply allow all possible sentences.

This then produces a new problem. How do we decide whether a sentence is true or false if nobody ever uses it?"

"Deuced difficult, what!"

"In fact the answer is forced on us by our insistence on a simple model. Let me show you. What have the following sentences got in common?

'Snow is white and a cube has 7 faces'.

'London is in England and a rabbit is not an animal'.

'The world is round and 5 is bigger than 12'."

"They all have the form

'a true sentence' and 'a false sentence'."

"Right. So in order to keep the model simple we insist that all sentences of this form behave in the same way, that is either they are all true or all false. We put a similar condition on other forms of sentences."

"Hold on! You still have to decide which they are all going to be, true or false."

"We will do that now. There are 14 different forms to consider, so I will leave you to work some out for yourself.

The first form is

not 'a true sentence'.

A typical example of this is

'not snow is white'."

"That is false and I would expect any sentence of this form to be false."

"So we put in our model that all sentences of this form are false. I will leave the sentences of the form

not 'a false sentence'

to you and also the 4 'and' sentences.

What about sentences of the form

'a true sentence' or 'a true sentence'?"

"Well, when I ask Babs if she wants to go to the Savoy or the Ritz I jolly well don't mean both. So I think these sentences are false."

"But what about that notice which says that you can get into the Test Match for half price if you are a student or a pensioner. You would make old Edgar Witherspoon pay full price just because he is attending evening classes during his retirement.

The trouble is that 'or' has two meanings and so to keep our model simple we select one and it happens to be the one which allows both parts of the sentence to be true. We select this meaning, so mathematicians use this one."

"So

'snow is white or a cube has 6 faces'

is true."

"That is correct. The other 3 'or' cases are quite straightforward and I will leave them to you. Finally we come to 'if-then'.

Let's take an example, say,

'if snow is white then a cube has 7 faces'."

"Well I would never start a sentence 'if snow is white then ...' since I know snow is white already."

"So you have to consider sentences where you don't know whether they are true or false. For example, do you know where your friend Algy is at the moment?"

"No."

"So what do you think of the sentence

'if Algy is in London, then Algy is in England'?"

"It is true."

"Good. Now consider the four cases:

'Algy is in London' is true and 'Algy is in England' is true,
'Algy is in London' is true and 'Algy is in England' is false,
'Algy is in London' is false and 'Algy is in England' is true,
'Algy is in London' is false and 'Algy is in England' is false.

Which are possible?"

"The 1st, 3rd and 4th are possible, the 2nd is not. In fact I think this is what I meant by saying the 'if-then' sentence was true."

"So you were assuming

if true then true,
if false then true,
if false then false

are all true and

if true then false

is false."

Then I went away and worked out the cases Atlas had left. When I returned I asked Atlas,

"How do we use these results?"

"Let us consider the case of bootlegging I am working on at the moment. The facts we have are these:

Al or Frank or Bugs or John was involved,
if Frank was involved then (Bugs was and Al was not),
if Johnny was involved then Bugs was not,
if Frank was not involved then (Johnny and Al were),
if Al was not involved then (Johnny was and Bugs was not).

All we have to do is to write down the list of possibilities and check each one against the facts. If we are lucky we will rule out all but one possibility and that will be our solution. If more than one possibility is left then we need more facts to rule out all but one of these possibilities which remain.

We appear to have 16 possibilities:

Al was involved.	Frank was involved.	Bugs was involved.	Johnny was involved
true	true	true	true
true	true	true	false
true	true	false	true

etc.

Let us take a typical possibility, say,

true false true false.

The first fact is a number of sentences joined by 'or'. You have worked out that when we have two sentences joined by 'or' the whole is true provided at least one of the two is true and false if they are both false. Well this extends to any number of sentences. So, as 'Al was invloved' is true the first fact is true.

The next fact begins 'if Frank was involved ...' and 'Frank was involved' is false and a sentence beginning 'if false then ...' is always true so the second fact is true. The same argument shows the third fact is true.

However the fourth fact begins 'if Frank was not involved ...' and 'not Frank was involved' is true so we have 'if true then ...' which means we have to look at the second half of the sentence Now 'Johnny and Al were' is only true if 'Johnny was involved' and 'Al was involved' are both true which is not the case. So we have 'if true then false' which is false so the fourth fact does not hold in this possibility so we can rule out this possibility. I will leave you to consider the other 15 possibilities and work out who was involved in the bootlegging.

By the way, what was the film you took the maid to see?"

"It was a story of police detection in San Francisco. You know the kind.

It is 1.00 a.m.

We are in the office at police headquarters. The lieutenant looks out of the window at the lights of the city spread out below.

'Somewhere out there is a man who has killed and will kill again unless we get to him first. But how can we find him?'

'There is one method that might work,' replies the chief, 'get out the telephone directories and pencils - lots of pencils'"

A little while after his triumph at Woodful Towers, Atlas remarked that he now thought me ready for a rather more formal introduction to the Propositional Calculus. He thereupon presented me with a manuscript which I reproduced in full.

Part 2.

The propositional calculus involves the symbols &(and), v(or), → (if ... then), ¬ (not), and letters p, q, r, s, ... which stand for sentences (such as 'snow is white', '5 is bigger than 12' etc.). The letters, in fact, perform the same function as the familiar x, y, z, ... do in algebra. Since, then, their job is to indicate places where a sentence may be put, they are called *sentence variables* or *sentence place-holders*. Our model is concerned with those rows of these symbols which become sentences when the sentence place-holders are replaced by sentences. Such a row is called a *well-formed formula* or *wff*.

Examples. (p & q), (¬ p v ¬ ¬(p → → q)) are wff's
p &, q v → rq ¬ are not wff's.

So far we have the part of the model concerned with structure. We now come to the part concerned with meaning. This involves two further symbols T (true) and F (false) called *truth values* and four evaluation lists called *truth tables*.

p	¬p
T	F
F	T

p	q	(p & q)
T	T	T
T	F	F
F	T	F
F	F	F

p	q	(p v q)
T	T	T
T	F	T
F	T	T
F	F	F

p	q	(p → q)
T	T	T
T	F	F
F	T	T
F	F	T

We have already seen (in Part 1) how these are arrived at.

The parts of the model are now assembled and we are ready to perform the basic calculations. Suppose we are given a wff and to each sentence place-holder in the wff is assigned a truth value. Then we can use the above truth tables to produce a single truth value in the way illustrated by the following example.

wff: ((p & q) v (¬(p & ¬¬q)→r)).

Assignment of truth values: T to p, F to q, F to r.

Replace the place-holders by the truth values:

$$((T \,\&\, F) \vee (\neg (T \,\&\, \neg\neg F) \rightarrow F)).$$

Look for any expressions which can be evaluated by the truth tables, $(T \,\&\, F) = F$, $\neg F = T$. Replace each of these expressions by the single truth value, so simplifying the whole expression:

$$(F \vee (\neg (T \,\&\, \neg T) \rightarrow F)).$$

Repeat the procedure. Since $\neg T = F$, we obtain

$$(F \vee (\neg (T \,\&\, F) \rightarrow F));$$

and further

$$(F \vee (\neg F \rightarrow F)),$$

$$(F \vee (T \rightarrow F)),$$

$$(F \vee F),$$

$$F.$$

Now we can think of a wff as a machine or function with inputs, in this case three, p, q, r. We input the assigned truth values and get out a truth value, in this case F. Because there is only a finite number of possible inputs we can list them and against each input give the resulting output. Thus we obtain a complete description of the wff working as a truth machine. We extend the use of the term 'truth table' to include such a list.

Example. The truth table for the wff $((p \,\&\, q) \vee (\neg (p \,\&\, \neg\neg q) \rightarrow r))$.

Input			Output
p	q	r	$((p \,\&\, q) \vee (\neg (p \,\&\, \neg\neg q) \rightarrow r))$
T	T	T	
T	T	F	
T	F	T	
T	F	F	F
F	T	T	
F	T	F	
F	F	T	
F	F	F	

The method used to obtain the given entry can be used to complete the table.

Now that we have the basic parts of the propositional calculus we can ask, how does it fit into the general field of mathematics and logic? The following points go some way to answering this question.

1. One can enlarge the model by adding parts to model the logic of the phrases 'for all ... such that' and 'there exists ... such that'. The new model is called the *restricted predicate calculus*.

2. The basic parts of the propositional calculus described above and the corresponding parts of the restricted predicate calculus form the working logic which the mathematician picks up incidentally in his training and which he uses every day to construct his sequences of mathematical deductions.

3. A similar remark applies to the computer programmer, although his use is more explicit and more formal and precise than that of the mathematician.

4. A lot of work is done in simply studying the model. This proceeds in the same way as any branch of pure mathematics by asking questions which seem interesting and trying to answer them. One important example of this is the adding to the model of a structure to model the idea of deduction. For example,

from 'if ΔABC is right-angled at A, then $AB^2 + AC^2 = BC^2$'
and 'ΔABC is right-angeld at A'
we deduce '$AB^2 + AC^2 = BC^2$'.

This is added to the model as the rule

$$\text{from } ((p \to q) \text{ and } p) \text{ deduce } q.$$

One can now take a selection of wff's and see what other wff's can be deduced from them. This is just the sort of thing that is done in axiomatic mathematics (e.g., group theory) where one begins with a set of axioms and aims to deduce interesting consequences.

To demonstrate the scope of the propositional calculus we discuss one of its important theorems and a rather surprising deduction from it.

Suppose we are trying to find a positive integer x to satisfy all of the following sequence of conditions,

$$x \neq 1, \quad x \neq 2, \quad x \neq 3, \quad x \neq 4, \ldots$$

If we take any finite set of the conditions, e.g.,

$$x \neq 2, \quad x \neq 5, \quad x \neq 13,$$

then we can find an x, say 1, to safisfy all the conditions in the set. However, when we put all the infinite number of conditions together we find there is no x which satisfies them all. Thus we have a situation where every finite part is satisfiable but the infinite whole is not.

Can we have a similar situation with the propositional calculus? Suppose we have an infinite sequence of wff's, e.g.,

$$(p_1 \mathbin{\&} p_3), \quad \neg p_2, \quad (p_4 \rightarrow \neg p_1), \quad (p_1 \mathbin{\&} p_1) \vee p_7, \ \ldots$$

which involves the sentence place-holders $p_1, p_2, p_3, \ldots$ (of which there may be an infinite number). We have to try to assign truth values, T or F, to the p_i's so as to give every wff in the sequence the truth value T. The *compactness theorem of the propositional calculus* says that if, for every finite set to wff's from the sequence we can find suitable truth values, then we can find suitable truth values for the whole infinite sequence.

The proof of this theorem goes roughly as follows. If we do an exhaustive search for possible suitable truth values

$$\text{for } p_1, \quad \text{for } p_1 \text{ and } p_2, \quad \text{for } p_1, p_2 \text{ and } p_3, \ \ldots$$

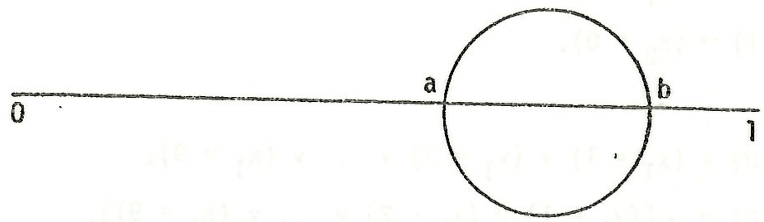
in turn, either we will succeed in finding suitable truth values for the whole infinite sequence or we will be stopped at some point in our search. This point will provide us with a finite set of wff's which cannot all be given the truth value T simultaneously. This situation contrasts with the above situation involving integers. The reason for the difference is that for all x there was an infinite number of possible choices of value, viz. 1, 2, 3, 4, ... whereas for each p_i there is only a finite number, viz. T, F.

The compactness theorem may be used to obtain a result about sets (which is usually called the Heine-Borel theorem). This may be stated in the following form. A line, one metre long, is covered by an infinite number of coins. (These coins may be of varying size and some of them may be very small.) It is then possible to remove all but a finite number of the coins and still have the line covered. By the way, when we talk of a point of the line being covered, we mean that the point lies underneath the interior of the coin, not just its rim.

To apply the compactness theorem we list the conditions which a point, say x, of the line would have to satisfy if it was not to be covered by the original coins.

To do so we express x as an infinite decimal $x_0.x_1x_2x_3 \ldots$; each of $x_1, x_2, x_3, \ldots$ takes one of the values $0, 1, \ldots, 9$; x_0 is 0 or 1 and if $x_0 = 1$, then $x_1, x_2, x_3, \ldots$ are all 0.

Suppose a coin is placed so



where a,b are the decimals $0.a_1a_2a_3 \ldots$, $0.b_ab_2b_3 \ldots$ respectively. If x is not covered by this coin, then we have the conditions

$$\{x_0 = 1\} \vee \{x_1 \leqslant a_1\} \vee \{x_1 \geqslant b_1\}$$

$$\{x_0 = 1\} \vee \{x_1 \leqslant a_1\} \vee \{(x_1 = b_1) \to (x_2 \geqslant b_2)\},$$

$$\{x_0 = 1\} \vee \{x_1 \leqslant a_1\} \vee \{[(x_1 = b_1) \ \& \ (x_2 = b_2)] \to (x_3 \geqslant b_3)\},$$

$$\{x_0 = 1\} \vee \{x_1 \leqslant a_1\} \vee \{[(x_1 = b_1) \ \& \ (x_2 = b_2) \ \& \ (x_3 = b_3)] \to (x_4 \geqslant b_4)\},$$

$$\vdots$$

$$\{x_0 = 1\} \vee \{(x_1 = a_1) \to (x_2 \leqslant a_2)\} \vee \{x_1 \geqslant b_1\},$$

$$\{x_0 = 1\} \vee \{(x_1 = a_1) \to (x_2 \leqslant a_2)\} \vee \{(x_1 = b_1) \to (x_2 \geqslant b_2)\},$$

$$\{x_0 = 1\} \vee \{(x_1 = a_1) \to (x_2 \leqslant a_2)\} \vee \{[(x_1 = b_1) \ \& \ (x_2 = b_2)] \to (x_3 \geqslant b_3)\},$$

and so on.

Repeat this for each coin and put together all the conditions on x we so obtain. (The conditions are simpler if $a < 0$ or $b > 1$.) Next replace each inequality as in the following example:

Replace $(x_4 \geqslant 6)$ by $[(x_4 = 6) \vee (x_4 = 7) \vee (x_4 = 8) \vee (x_4 = 9)]$.

Then add to the conditions the following which ensure that each x_i is given exactly one value and that the special case of $x_0 = 1$ is taken care of:

$$(x_0 = 0) \vee (x_0 = 1),$$
$$(x_0 = 0) \rightarrow \neg (x_0 = 1),$$
$$(x_0 = 1) \rightarrow \neg (x_0 = 0),$$
$$(x_0 = 1) \rightarrow (x_1 = 0),$$
$$(x_0 = 1) \rightarrow (x_2 = 0),$$
$$\vdots$$
$$(x_1 = 0) \vee (x_1 = 1) \vee (x_1 = 2) \vee \ldots \vee (x_1 = 9),$$
$$(x_1 = 0) \rightarrow \neg [(x_1 = 1) \vee (x_1 = 2) \vee \ldots \vee (x_1 = 9)],$$
$$(x_1 = 1) \rightarrow \neg [(x_1 = 0) \vee (x_1 = 2) \vee \ldots \vee (x_1 = 9)],$$
$$\vdots$$
$$(x_1 = 9) \rightarrow \neg [(x_1 = 0) \vee (x_1 = 1) \vee \ldots \vee (x_1 = 8)],$$
$$(x_2 = 0) \vee (x_2 = 1) \vee (x_2 = 2) \vee \ldots \vee (x_2 = 9),$$
$$(x_2 = 0) \rightarrow \neg [(x_2 = 1) \vee (x_2 = 2) \vee \ldots \vee (x_2 = 9)],$$

and so on.

We now have a complete set of conditions on x and they are made up of $\&$, $\vee$, $\rightarrow$, $\neg$ and the equations

$$(x_0 = 0),\ (x_0 = 1),\ (x_1 = 0),\ (x_1 = 1),\ (x_1 = 2),\ \ldots,\ (x_1 = 9),\ (x_2 = 0),\ (x_2 = 1)\ \ldots$$

Replaces these equations by the sentence place-holders

$$p_1,\ p_2,\ p_3,\ p_4,\ p_5,\ \ldots,\ p_{12},\ p_{13},\ p_{14},\ \ldots$$

respectively. The set of conditions becomes a set of wff's. Then finding an x to satisfy the conditions is equivalent to assigning truth values, T or F, to the p_i so as to give all the wff's the truth value T.

It follows from the compactness theorem that if we cannot find an uncovered point x then there is a finite set of the conditions which cannot be satisfied. This finite set must have come from some finite set of coins and these coins must cover the line.

Further reading

A. Basson and D. O'Connor, Introduction to Symbolic Logic (University Tutorial Press, 1959).