SOLUTIONS TO THE 1982 SCHOOL MATHEMATICS COMPETITION RUN BY U.N.S.W. AND SPONSORED BY IBM.

JUNIOR DIVISION

1. Suppose we enter a number, abc say, in a calculator and then repeat it to get abcabc. Now divide by 13, 11 and 7 consecutively. The calculator will always display the original number abc; why?
   Are there any numbers so that if we divide abab by them, we always get ab?
   What about abcdabcd?

ANSWER

abcabc = abc x 1001 and 1001 = 7.11.13. abab ÷ 101 = ab and abcdabcd ÷ 10001 = abcd.

Hence all we have to do is check for prime factors of 101 and 10001.

The primes less than 100 are 2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59, 61,67,71,73,79,83,89 and 97. Hence 101 is prime and (using a calculator) 10001 = 73 x 137 which are both prime. The answer is thus 101 to the second question and 10001 or 73 followed by 137 to the third question.

2. Find the next statement in the pattern

$$5^3 + 8^3 = 13.7^2$$

$$7^3 + 15^3 = 22.13^2$$

$$9^3 + 24^3 = 33.21^2$$

What is the general rule?

ANSWER

The next numbers are

5, 7, 9	11,13,15
8,15,24	35,48,63
13,22,33	46,61,78
7,13,21	31,43,57

and $11^3 + 35^3 = 46.31^2$

$$13^3 + 48^3 = 61.43^2$$

$$15^3 + 63^3 = 78.57^2$$

which can be checked on a calculator to be true.

The pattern 5,7,9,11,13,15 is $2n+1$ with $n = 2,3,4,5,6,7$. The other sequences are all quadratic functions of n. Suppose the sequence 8,15,24,35, 48,63 is given by $an^2 + bn + c$ with $n = 2,3,\ldots,7$ then

$$4a + 2b + c = 8$$

$$9a + 3b + c = 15$$

$$16a + 4b + c = 24$$

$$5a + b = 7$$

$$7a + b = 9$$

$a = 1$, $b = 2$ and $c = 0$.

i.e. the sequence is given by

$$n^2 + 2n$$

Similarly the sequence 13,22,33,46,61,78 is $n^2 + 4n + 1$, $n = 2,3,4,5,6,7$ and the sequence 7,13,21,31,43,57 is $n^2 + n + 1$ with $n = 2,3,4,5,6,7$.

Hence the pattern is

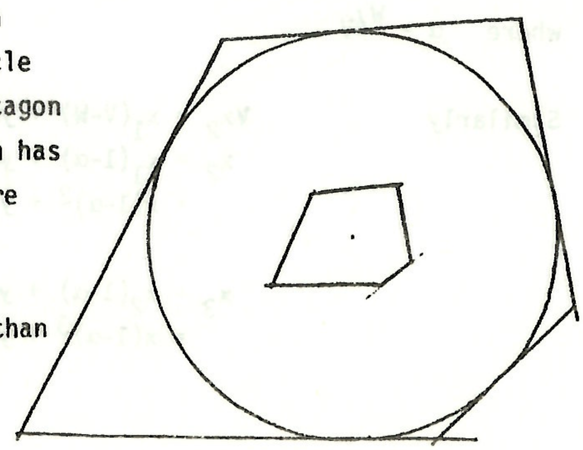
$$(2n+1)^3 + (n^2+2n)^3 = (n^2+4n+1)(n^2+n+1)^2$$

which can be checked by expanding both expressions, or by checking for 6 distinct values of n.

3. In an arbitrary convex pentagon each side is translated outwards by 4 units. Prove that the area is increased by at least 50 square units.

ANSWER

If we replace a pentagon by a smaller pentagon of exactly the same shape then the amount by which the area increases is less. Hence we simply have to prove the result for a small pentagon (with arbitrary angles.) If we draw a circle with centre any point inside the pentagon and radius 4 then the larger pentagon has area greater than $\pi 4^2 = 50.265$ square units. Hence if we make the smaller pentagon smaller than ¼ square unit. the area has been increased by more than



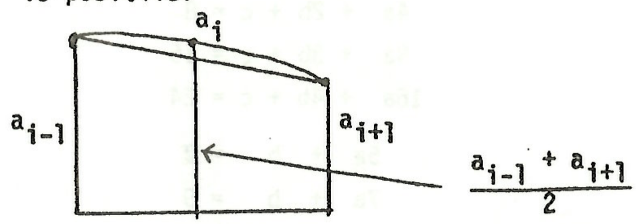
50 square units. (The best result possible is 80 tan 36^0 = 58.123 square units).

4. Among the numbers $a_1, a_2, \ldots, a_{1982}$ the first and last one are positive and

$$a_{i-1} + a_{i+1} \leqslant 2a_i \quad \text{for} \quad 1 < i < 1982.$$

Prove that each a_i is positive.

ANSWER



The sequence is "convex" as in the above diagram. Suppose $a_1 \ldots, a_{i-1}$ are positive and a_i is negative. Then clearly $a_{i+1}, a_{i+2}, \ldots$ are all negative by convexity. However $a_{1982} > 0$, hence $a_i > 0$ for all i.

5. A large bottle contains an x% solution of acid. A volume of W litres is drawn off, replaced by the same quantity of a y% solution of the acid and thoroughly mixed. The operation is repeated n times until a z% solution is obtained. Find the volume of the bottle, V litres, in terms of W,n,x,y and z. (Hint: It may help to first solve the problem when n = 3.)

ANSWER

Let the solution be x_1, x_2,...% after one, two,... repetitious. Then

$$Vx_1 = x(V-W) + yW$$
$$x_1 = x(1-\tfrac{W}{V}) + y\tfrac{W}{V} = x(1-\alpha) + y\alpha$$

where $\alpha = W/V$

Similarly

$$Vx_2 = x_1(V-W) + yW$$
$$x_2 = x_1(1-\alpha) + y\alpha$$
$$= x(1-\alpha)^2 + y(1-\alpha)\alpha + y\alpha$$

$$x_3 = x_2(1-\alpha) + y\alpha$$
$$= x(1-\alpha)^3 + y\alpha(1-\alpha)^2 + y\alpha(1-\alpha) + y\alpha$$

Hence $z = x_n = x(1-\alpha)^n + y\alpha(1-\alpha)^{n-1} + \dots + y\alpha(1-\alpha) + y\alpha$.

The difficult part of the problem is to make α the subject of this equation.

$(1-\alpha)z = x(1-\alpha)^{n+1} + y\alpha(1-\alpha)^n + \dots + y\alpha(1-\alpha)^2 + y\alpha(1-\alpha)$ and subtracting the two equations from each other

$$z - z + \alpha z = x(1-\alpha)^n - x(1-\alpha)^{n+1} + y\alpha - y\alpha(1-\alpha)^n$$

$$\alpha(z-y) = x(1-\alpha)^n(1-1+\alpha) - y\alpha(1-\alpha)^n$$

$$(1-\alpha)^n = \frac{z-y}{x-y}$$

$$1-\alpha = \sqrt[n]{\frac{z-y}{x-y}}$$

$$\frac{W}{V} = \alpha = 1 - \sqrt[n]{\frac{z-y}{x-y}}$$

$$V = \frac{W}{1 - \sqrt[n]{\frac{z-y}{x-y}}}.$$

6. A cub reporter interviewed four people. He was very careless, however. Each statement he wrote was half right and half wrong. He went back and interviewed the people again. And again, each statement he wrote was half right and half wrong. From the information below, can you straighten out the mess?

The first names were Jane, Len, Mary and Paul. The last names were Gray, King, Brown and Hunt. The ages were 32, 38, 45 and 55. The occupations were teacher, pilot, police sergeant, and driving instructor.

On the first interview, he wrote these statements, one from each person:

1. Jane: "My name is Gray, and I'm 45."
2. King: "I'm Paul and I'm a driving instructor."
3. Len: "I'm a police sergeant and I'm 45."
4. Hunt: I'm a teacher, and I'm 38."

On the second interview, he wrote these statements, one from each person:

5. Brown: "I'm a pilot, and my name is Len."
6. Jane: "I'm a pilot, and I'm 45".
7. Mary: "I'm 55 and I'm a driving instructor."
8. Hunt: "I'm 38 and I'm a driving instructor."

ANSWER

Since the 4 people are called, Jane, King, Len and Hunt and also called Brown, Jane, Mary and Hunt it follows that Jane is not King, Hunt or Brown and that Hunt is not Jane, Len or Mary. Hence Jane is Jane Gray and Hunt is Paul Hunt. Hence Mary is Mary King and Len is Len Brown.

Now Mary King is not Paul, thus she is a driving instructor. Jane Gray is Gray and thus not 45 and thus is a pilot. Next Paul Hunt is not the driving instructor, hence he is 38. Hence he is not the teacher, nor a pilot, and thus Hunt is a police sergeant. Hence Len Brown is the teacher and he is 45. Mary King is neither 55, 38 nor 45 thus she is 32. Finally Jane Gray is 55.

Len Brown	45	Teacher
Jane Gray	55	Pilot
Paul Hunt	38	Police Sergeant
Mary King	32	Driving instructor

SENIOR DIVISION

1. Andrew, Sarah, and Kim live in different Sydney suburbs (Burwood, Parramatta, Hornsby) and were each doing something different (sitting at home, eating in a restaurant, working) when a power failure struck. Each used something different for light (candles, torch, kerosene lamp) until power was restored. The last names of the people are Smith, Cox, and Taylor.

   Match up each person's full name with the suburb, the temporary source of light, and the place (home, restaurant, work) where the person was at the time of the power failure.

   1. The person who lives in Burwood, Sarah's sister, did not use candles during the power failure.
   2. The man who used a torch took longer than the 45 minutes Smith took to get home when the power failed.
   3. The person who lives in Hornsby was glad she was not at work when the power failed.
   4. Sarah, who was not at home when the power failed, called Cox to see how he was.

ANSWER

1. implies Kim is Female. Hence, from 2., Andrew used the torch, Andrew is not Andrew Smith and he was not at home. 1. implies Kim lives in Burwood and she did not use candles. 3. implies that Sarah lives in Hornsby and was not at work. 4. implies Sarah was also not at home, hence she was at the Restaurant. 4. implies Andrew is Andrew Cox and 2. implies he was at work. Kim lives in Burwood, Sara lives in Hornsby hence Andrew lives in Parramatta. Andrew was at work, Sarah was at the restaurant, hence Kim is at home. Smith was not at home and Kim is not Kim Cox hence Kim is Kim Taylor and Sarah is Sarah Smith, Andrew used the torch, Kim did not use candles, hence Kim used the kerosene lamp and Sara used candles.

Andrew Cox	Parramatta	Work	torch
Sarah Smith	Hornsby	Restaurant	candles
Kim Taylor	Burwood	home	kerosene lamp.

2. Two play the following game: they take turns in naming a positive divisor of 1000 with the condition that they cannot name a divisor which divides a number already named. The one who names 1000 loses. Prove that if the starter plays correctly, he will win.

What happens if the game is modified so that no number can be named which has fewer divisors than any of the numbers previously named?

What happens if 1000 is replaced by 27000 in the first version?

ANSWER

The first player names 100 and then replies 8 to 125, 40 to 250, 200 to 500, 125 to 8, 250 to 40 and 500 to 200 when necessary. Then the second player is forced, finally, to name 1000.

The second game is simpler as the first player names 100 and the only possible responses are 200, 500 or 1000. 200 is followed by 500 then 1000. 500 is followed by 200 then 1000.

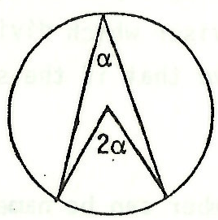
I cannot find an explicit solution to the third game. However, the following sentence is a proof that the first player will win if he plays correctly. Either the first player can select a divisor other than 1 and win or all such selections lead to losses and the first player wins by selecting 1 and forcing the second player to choose a losing divisor!!

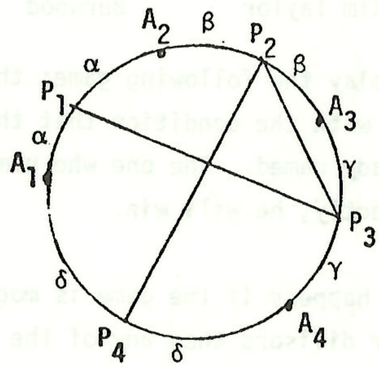
3. The circumference of a circle is divided into four arcs. Show that two of the line segments joining the mid-points of these arcs are at right angles to each other.

ANSWER

The only theorem needed is that the angle subtended by an arc at the centre of a circle is twice the angle subtended by the arc of the circumference of the circle.

Hence if we label the arcs of a circle by the angles subtended at the centre of a circle then the sum of the labels is 360°.





Let the 4 arcs be A_1A_2, A_2A_3, A_3A_4 and A_4A_1 with midpoints P_1, P_2, P_3, P_4 and labels 2α, 2β, 2γ and 2δ.

Now the arc P_1P_2 subtends $\alpha+\beta$ at the centre, hence $P_1P_3P_2$ is $\frac{1}{2}(\alpha+\beta)$.

Similarly $P_4P_2P_3$ is $\frac{1}{2}(\gamma+\delta)$. Now $\frac{1}{2}(\alpha+\beta+\gamma+\delta)$ 90° and

$$P_2 X P_3 = 180^{\circ} - \tfrac{1}{2}(\alpha+\beta) - \tfrac{1}{2}(\gamma+\delta)$$
$$= 180^{\circ} - 90^{\circ} = 90^{\circ}$$

and P_1P_3 meets P_2P_4 at right angles.

4. The function f maps the interval $I = \{x: 0 \leqslant x \leqslant 1\}$ into itself, that is $f(x)$ is defined for $0 \leqslant x \leqslant 1$ and

$$0 \leqslant f(x) \leqslant 1 \text{ for all } x \text{ in } I.$$

Furthermore, $f(x)$ satisfies

$$f(x) + f(y) = f(f(x) + y)$$

for all x, y for which both sides of the equation are defined.

Prove that $f(f(x)) = f(x)$ for all x in I.

ANSWER

If one puts $y = 0$ then

$$f(x) + f(0) = f(f(x) + 0) = f(f(x))$$

Hence we only have to prove $f(0) = 0$.

Now $f(0) + f(0) = f(f(0))$. Thus $f(0) \leq \frac{1}{2}$

Let

$$f_1(0) = f(0) \quad \text{and} \quad f_{i+1} = f(f_i(0)).$$

$$\text{i.e. } f_3(0) = f(f(f(0))).$$

Now

$$\begin{aligned} f_3(0) &= f(f_2(0)). \\ &= f(f_2(0) + 0) \\ &= f_2(0) + f(0) \\ &= f(0) + f(0) + f(0). \end{aligned}$$

Similarly

$$\begin{aligned} f_{i+1}(0) &= f(f_i(0) + 0) = f_i(0) + f(0) \\ &= (i+1)f(0) \end{aligned}$$

However, $f_i(0) \leq 1$ thus $f(0) \leq 1/i$ for all $i = 1,2,3,\ldots$. Hence $f(0) = 0$ and

$$f(f(x)) = f(x)$$

5. A set of rational numbers, T has the following properties:

(i) If $\frac{p}{q}$ is in T then also $\frac{p}{p+q}$ and $\frac{p}{p+q}$ are in T.

(ii) $\frac{1}{2}$ is in T.

Prove that T contains all rational numbers between 0 and 1.

ANSWER

Since $\frac{1}{2} \in T$, (i) implies that $1/1+2 = 1/3 \in T$ and $2/1+2 = 2/3 \in T$. Hence 1/4, 3/4, 2/5, 1/5 and 4/5 all $\in T$.

We shall prove the result by induction on $p+q$ where the rational is p/q. The result is true if $p+q = 3,4,5$ or 6.

If $\frac{p}{q-p} < 1$ then $\frac{q}{(q-p)+p} = p/q \in T$

or if $\frac{q-p}{p} < 1$ then $\frac{p}{(q-p)+p} = p/q \in T$.

since $p+(q-p) = q < p+q$ and by induction the result is proved.

6. Find all solutions in positive integers x,y of the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{3}{1982}.$$

ANSWER

Without loss of generality $y \geqslant x$ hence $660 < x < 1321$ and thus there are only a finite number of solutions. Now 991 is not divisible by 2,3,5,7,11, 13,17,19,23,29 or 31 and is thus prime and 1982 is completely factorized as 2.991.

$$\frac{3}{1982} - \frac{1}{x} = \frac{3x - 1982}{1982.x} = \frac{1}{x}$$

Suppose p is a prime divisor of 1982 and $p|3x - 1982$ then $p|3x$ and $p|x$. Conversely if $p|x$ and $p|3x - 1982$ then $p|1982$. Hence all factors of $3x - 1982$ are factors of x and 1982, which are 1,2,991 or 1982. Hence $3x - 1982$ divides $(1982)^2$ and lies between 1982. Hence $3x - 1982$ divides $(1982)^2$ and lies between 1 and 1981.

So

$3x - 1982 = 1,2,4,$ or 991

$3x = 1983,1984,1986$ or 2973

$x = 661,662$ or 991 giving

$y = 1982.661,981.331$ or 1982

respectively.