SOLUTIONS TO PROBLEMS FROM VOLUME 17, NUMBER 3.

Q. 503. A rectangle 11 cms × 7 cms is divided by ruled lines into 1 cm × 1 cm squares, each containing a button. Is it possible to replace all 77 buttons on the rectangle so that each one occupies a square adjacent to its original square? (Squares are adjacent if they share a common side.)

SOLUTION. Suppose the squares are coloured black and white as on a chessboard, with the corner squares all white. Let each button be of the same colour as the square on which it rests. Since there are 39 white buttons and only 38 black squares it is impossible that the buttons can be rearranged with all black buttons on white squares and all white buttons on black squares, as would be required if every button was moved to an adjacent square.

Correct Solution supplied by Andrew Jenkins (North Sydney Boys' High School).

Q. 504. Confirm that $2^{105} + 3^{105}$ is divisible by 7, 11, 25, and 463, but not by 13.

SOLUTION. If n is any odd whole number the following algebraic identity can be checked by multiplying out the R.H.S.:-

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 + \dots + (-1)^k x^{n-1-k}y^k + \dots + y^{n-1})$$

It follows that if x and y are integers, $x + y$ is a factor of $x^n + y^n$. Taking $x = 2^3$, $y = 3^3$, $n = 35$ we see that $5 \times 7 = 2^3 + 3^3$ is a factor of $(2^3)^{35} + (3^3)^{35} = 2^{105} + 3^{105}$. Similarly $11 \times 25 = 2^5 + 3^5$, and $5 \times 463 = 2^7 + 3^7$ are factors of the same number ($n = 21$ and 15 respectively.)

To show that 13 is not a factor using the same approach, one could observe that $13 = 2^2 + 3^2$ is a factor of $(2^2)^{53} + (3^2)^{53} = 2^{106} + 3^{106}$. If it were also a factor of $2^{105} + 3^{105}$ then it would be a factor of $(2^{106} + 3^{106}) - 2(2^{105} + 3^{105}) = (3 - 2).3^{105} = 3^{105}$. But by the uniqueness of factorisation into primes, it is clear that 3^{105} is not divisible by 13. Q.E.D.

Correct solution from R. Bozier (Barker College).

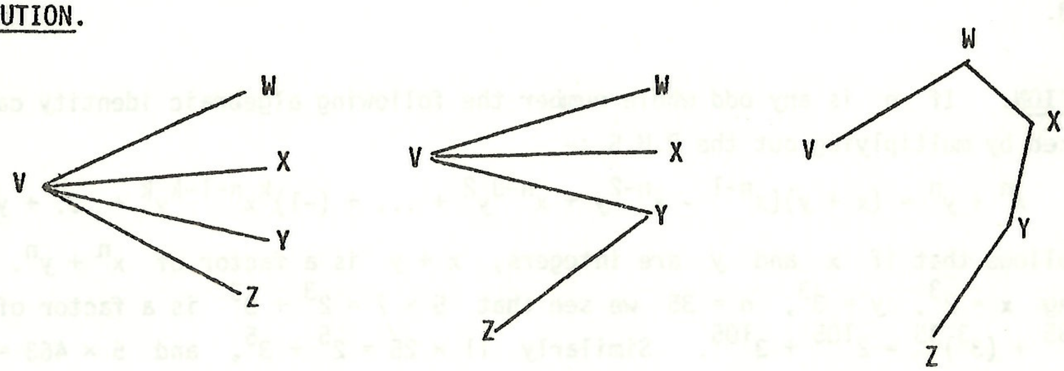
Q. 505. You are given $3n$ points in the plane no three of which are collinear. Is it always possible to draw n triangles having the points as vertices, in such a way that no two triangles overlap? Either prove it, or produce a counter example.

SOLUTION. Choose a direction not parallel to any of the line segments determined by two of the points. Start with a line in this direction with all the points on one side. Move the line across the plane until it has passed over three of the points. Draw the triangle determined by the three points. Since the remaining $(3n - 3)$ points all lie on the other side of the present position of our moving line no triangle with three of those points as vertices will overlap the triangle we have already drawn. The line can now be again moved parallel to itself until another 3 points have been passed over, determining a second triangle. Obviously the process can be repeated to obtain n triangles which do not overlap.

Correct solution from A. Jenkins (North Sydney Boys' High School).

Q. 506. A spider has 5 hide-outs at points A, B, C, D and E in a tree, no four of them lying in a plane. In how many different ways could the spider connect them together by just four straight threads of silk, each ending at two of the points.

SOLUTION.



There are 3 essentially different patterns by which the hide-outs could be connected. (See figures above.)

In pattern (1), one hide-out V is joined to each of the others. There are five different ways of choosing V from A, B, C, D and E, but thereafter no further choice is required. Hence there are 5 ways of joining the hide-outs using (1).

In pattern (2), V is joined to three of the other hide-outs. The last hide-out, Z, is then joined to one of those three. There are 5 ways of choosing V; then for each choice of V there remain 4 ways of choosing Z. Finally there are 3 ways to choose which of W, X, Y to join to Z. Hence there are $60 = 5 \times 4 \times 3$ ways of joining the hide-outs using (2).

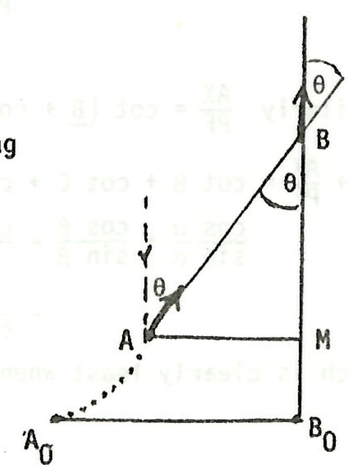
In pattern (3), no hide-out is joined to more than 2 others. There are 5! ways of ordering the 5 symbols A, B, C, D, E, but only $60 = \frac{1}{2} . 5!$ different ways for the spider to join the hide-outs using (3), since each way of joining the hide-outs as in the figure corresponds to the two orderings V W X Y Z and Z Y X W V.

Hence altogether the spider can connect his hideouts in $5 + 60 + 60 = 125$ different ways.

Q. 507. Let us assume that in a knockout tennis tournament the players always play true to form, so that the stronger player invariably defeats a weaker opponent. If there are 16 entrants in the tournament, and the draw is decided by lot, what is the probability that the final (in round 5) is played between the strongest two players?

SOLUTION. Whichever of the 16 places in the draw is allotted to the strongest player A, the remaining 15 places are equally likely for the second strongest B. Of these 8 are in the opposite half of the draw to A, and 7 in the same half. Thus the probability that A meets B in the final is $\frac{8}{15}$.

Q. 508. A pirate ship A sees a ship B 20 miles due East sailing due North. A gives chase by always sailing directly towards B. However both ships have the same speed, so that A eventually is also sailing virtually northwards, some distance behind B. How far behind?



SOLUTION. The diagram shows the positions A, B, of the two ships at some stage of the chase. The starting positions were the points A_0 and B_0; M is the point due East of A on B_0 B. We shall show that the instantaneous rate at which the distance A B is decreasing is exactly the same as the rate at which B M is increasing. It then follows that the sum of the distances AB + BM remains unchanged throughout the chase, at the starting value 20 + 0 miles. Eventually when A and M are virtually the same point, we must have AB = MB = 10 miles.

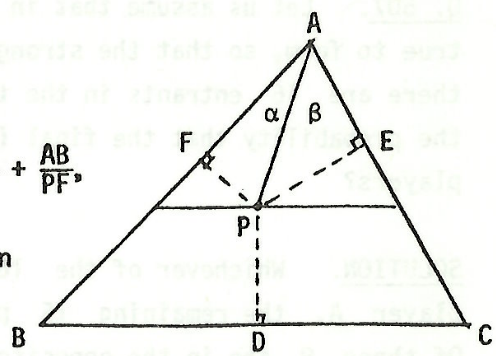
Let the speed of both ships be v m.p.h, and let θ be the angle ABM. The northward component of A's speed is then $v \cos\theta$, and this is M's speed northward. Hence BM is increasing in length at a speed of $(v - v\cos\theta)$.

In the AB direction, B's component of speed is $v\cos\theta$, so that AB is decreasing in length at A's speed - B's speed in this direction viz $v - v\cos\theta$. Hence these two rates are equal, as promised.

Q. 509. P is a point inside a given triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Find all P for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is least.



SOLUTION. Let P be a minimum point of $\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$, and construct a line XY through P parallel to BC. If P is moved along XY, the first term in the expression does not change, so the minimum position is also the minimum point in this line for $\frac{CA}{PE} + \frac{AB}{PF}$, and of $\frac{YA}{PE} + \frac{AX}{PF}$ since $\frac{YA}{CA} = \frac{AX}{AB}$. We will show that the minimum value of $\frac{YA}{PE} + \frac{AX}{PF}$ occurs when AP bisects $\angle XAY$. Similarly we can prove that PB bisects $\angle ABC$, whence the only minimum point P is the incentre of the triangle. Now

$$\frac{YA}{PE} = \frac{YE}{PE} + \frac{EA}{PE} = \cot \angle EYP + \cot \angle EAP$$
$$= \cot \angle C + \cot \beta \quad \text{(see Figure)}$$

Similarly $\frac{AX}{PF} = \cot \angle B + \cot \alpha$. It follows that the minimum value of

$\frac{YA}{PE} + \frac{AX}{PF} = \cot B + \cot C + \cot \alpha + \cot \beta$ occurs when $\cot\alpha + \cot\beta$ is least

Now $$\frac{\cos\alpha}{\sin\alpha} + \frac{\cos\beta}{\sin\beta} = \frac{\cos\alpha \sin\beta + \cos\beta \sin\alpha}{\sin\alpha \sin\beta} = \frac{2\sin(\alpha+\beta)}{\cos(\alpha-\beta) - \cos(\alpha+\beta)}$$
$$= \frac{2\sin A}{\cos(\alpha-\beta) - \cos A}$$

which is clearly least when $\alpha - \beta = 0$; i.e. when AP bisects $\angle A$.

Q. 510. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Also consider the least number in each of these subsets. $F(n,r)$ denotes the arithemtic mean of these least numbers; prove that

$$F(n,r) = \frac{n+1}{r+1}.$$

SOLUTION. Of the ^nC_r subsets, the number having 1 as the smallest element is $^{n-1}C_{r-1}$, since such subsets in addition to 1 contain a further $(r-1)$ elements chosen from $\{2, 3, \ldots, n\}$. Similarly $^{n-2}C_{r-1}$ of them have 2 as the smallest element, in fact $^{n-k}C_{n-1}$ have k as the smallest element, for $k = 1, 2, \ldots, n-r+1$. We observe from this first that

$$^nC_r = {}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + {}^{n-3}C_{r-1} + \ldots + {}^{r-1}C_{r-1} \quad (1)$$

and secondly that

$$F(n,r) = \frac{^{n-1}C_{r-1}\,1 + {}^{n-2}C_{r-1}\,2 + {}^{n-3}C_{r-1}\,3 + \ldots + {}^{n-2}C_{r-1}(n-r+1)}{^nC_r} \quad (2)$$

We can use the identity (1) (which is true for all whole numbers n and r for which $n \geqslant r$), to show that the numerator of the R.H.S. of (2) is $^{n+1}C_{r+1}$. In fact the numerator is equal to

$$({}^{n-1}C_{r-1} + {}^{n-2}C_{r-1} + \ldots + {}^{r-1}C_{r-1}) + ({}^{n-2}C_{r-1} + {}^{n-3}C_{r-1} + \ldots + {}^{r-1}C_{r-1}) + \ldots$$
$$+ ({}^{n-k}C_{r-1} + {}^{n-k-1}C_{r-1} + \ldots + {}^{r-1}C_{r-1}) = \ldots + ({}^{r-1}C_{r-1})$$
$$= {}^nC_r + {}^{n-1}C_r + \ldots + {}^{n-k+1}C_r + \ldots + {}^rC_r$$
$$= {}^{n+1}C_{r+1}.$$

Hence

$$F(n,r) = \frac{^{n+1}C_{r+1}}{^nC_r} = \frac{\dfrac{(n+1)!}{(r+1)!(n-r)!}}{\dfrac{n!}{r!(n-1)!}} = \frac{n+1}{r+1}.$$

Q. 511. Determine the maximum value of $m^2 + n^2$, where m and n are integers satisfying m,n $\{1, 2, \ldots, 1981\}$ and $(n^2 - mn - m^2)^2 = 1$.

SOLUTION. Let $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. This generates the

Fibonacci sequence $\{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots\}$. We claim that all pairs (m,n) with $m,n \in \{1, 2, 3, \ldots, 1981\}$ which satisfy $n^2 - mn - m^2 = \pm 1$ are given by $(m,n) = (F_k, F_{k+1})$ for $k = 1, \ldots, 16$. It will of course then follow that the maximum value of $m^2 + n^2$ is equal to $987^2 + 1597^2$.

To validate our claim notice that if $(x,y) = (m,n)$ satisfies

$$y^2 - xy - x^2 = \pm 1 \tag{1}$$

with m and n both positive integers then

a) $m \leqslant n \leqslant 2m$ and either of m,n determines the other uniquely.

b) $(x,y) = (n,m+n)$ is another solution in positive integers

c) $(x,y) = (n-m,m)$ is another solution in non-negative integers

To prove (a) observe that regarded as a quadratic in either x or y, the product of the roots $(-y^2 \pm 1$, or $-x^2 \pm 1)$ is never positive (for positive integer values of y,x). Hence there cannot be two positive integer roots. Since $y^2 - xy - x^2$ changes sign from $-m^2$ to $+m^2$ on changing (x,y) from (m,m) to $(2m,m)$, it follows that n must lie in the interval $[m,2m]$.

(b) follows from $(m + n)^2 - n(m + n) - n^2 = -[n^2 - mn - m^2]$.

(c) follows from $m^2 - (n - m)m - (n - m)^2 = -[n^2 - mn - m^2]$, together with $m \leqslant n$ which ensures that $n - m$ is not negative. Indeed since the only solution with m equal to n is $(1,1)$, $n - m$ is positive for values of n larger than 1.

Now it can be checked immediately that $(x,y) = (F_1, F_2) = (1,1)$ satisfies (1). Assume that for some k, $(x,y) = (F_k,F_{k+1})$ is a solution. Then, by (b), so is $(F_{k+1},F_k + F_{k+1}) = (F_{k+1},F_{k+2})$ By Mathematical induction (F_k,F_{k+1}) is a solution for all k. It remains only to check that there are no other solutions of (1). Suppose there are other solutions and let $(x,y) = (m,n)$ be that with the smallest value of y. Then by (c) $(n-m,m)$ is a solution of (1) with a smaller value of y. Hence it must be of the form (F_k,F_{k+1}) for some k. But if $n - m = F_k$ and $m = F_{k+1}$ then $n = F_k + F_{k+1} = F_{k+2}$, so that (m,n) was after all in the given list of solutions, a contradiction

<u>Q. 512</u>. a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n - 1$ numbers?

b) For which values of $n > 2$ is there exactly one set having the stated property?

SOLUTION. Let L be the largest of a block of n consecutive positive integers. If the factorisation of L into primes is $L = 2^{\alpha_1} \times 3^{\alpha_2} \times \ldots \times p_k^{\alpha_k} \ldots$ where p_k denotes the kth prime, (all save a finite set of the powers α_k are zero), then in order for the stated condition to be satisfied it is necessary and sufficient that $L \geqslant n$ and that $p_k^{\alpha_k} < n$ for every k.

Indeed, if for any k, $p_k^{\alpha_k} \geqslant n$, then none of the numbers $L - 1, L - 2, \ldots, L - n + 1$, contains $p_k^{\alpha_k}$ as a factor, (since the nearest multiple of $p_k^{\alpha_k}$ below L is $L - p_k^{\alpha_k}$) and their l.c.m. cannot be divisible by L. In the other direction, if every $p_k^{\alpha_k}$ is less than n then the numbers $L - 2^{\alpha_1}, L - 3^{\alpha_2}, \ldots, L - p_k^{\alpha_k}, \ldots$ are all included amongst $L - 1, L - 2, \ldots, L - n + 1$, and have factors $2^{\alpha_1}, 3^{\alpha_2}, \ldots, p_k^{\alpha_k}, \ldots$ respectively. Thus their l.c.m. is divisible by L.

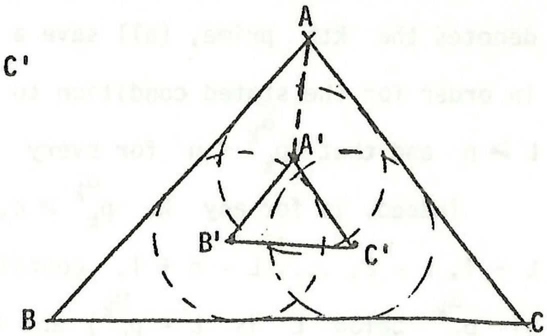
If $n = 3$, the largest possible value of $p_k^{\alpha_k}$ is 2^1. The only values for L are 2^0 or 2^1 neither of which is as large as 3. Hence there is no solution for $n = 3$.

If $n = 4$, $p_k^{\alpha_k}$ can be 2^1 or 3^1. Of the numbers L having factors 2^0 or 2^1, and 3^0 or 3^1 only one is as large as 4, viz $L = 2 \times 3 = 6$. Thus for $n = 4$ there is one set of 4 consecutive numbers having the required property viz. 3, 4, 5, 6.

If $n = 5$, $p_k^{\alpha_k}$ can be as large as 2^2 or 3^1. The numbers $L = 2 \times 3$ and $2^2 \times 3$ both exceed 5 and head sets of consecutive integers with the required property.

If $n = 6$, $p_k^{\alpha_k}$ can be as large as 2^2, 3^1, or 5^1. and L can be any one of 6, 10, 12, 15, 20, 30, or 60. It is clear that for larger values of n, there will be several possible values of L. For example if $2^t < n \leqslant 2^{t+1}$ and $6 \leqslant n$ then two possible values of L are $2^t \times 3$ and $2^t \times 5$.

Q. 513. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incentre and the circumcentre of the triangle, and the point O are collinear.



SOLUTION. In the figure let A', B', and C' denote the centres of the three circles, and ABC the given triangle. If the radius of each of the three circles is r then the distance of A' and B' from the side AB is r whence $A'B' \parallel AB$. Similarly $A'C' \parallel AC$ and $B'C' \parallel BC$. Since A' is equidistant from AB and AC it lies on the bisector of $\angle A$, and because of the parallels observed this line produced is also the bisector of $\angle A'$ in $\Delta A'B'C'$. Similarly for the angle bisectors of $\angle B$ and $\angle C$. Hence their point of intersection, I, is the incentre of both triangles ABC and $A'B'C'$. If O is the point common to the three circles then $OA = OB = OC = r$ whence O is the circumcentre of $\Delta A'B'C'$.

Note that ΔABC is similar to $\Delta A'B'C'$ and let λ be the magnification factor $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$. Produce IO to X where $\frac{IX}{IO} = \lambda$. We shall complete the proof by showing that X is the circumcentre of ΔABC. Indeed we shall show that the distance of X from each vertex of ΔABC is λr.

From the similar triangle ΔIAB and $\Delta IA'B'$.

$$\lambda = \frac{AB}{A'B'} = \frac{IA}{IA'} .$$

Now the triangles ΔIAX and $\Delta IA'O$ are seen to be similar, since $\frac{IA}{IA'} = \frac{IX}{IO} = \lambda$ and the included angle at I is common. Hence $\frac{AX}{A'O} = \lambda$ so that $AX = \lambda \times A'O = \lambda r$. Similarly $BX = CX = \lambda r$ and the proof is complete.

Q. 514. The function $f(x,y)$ satisfies

1. $f(0,y) = y + 1$,
2. $f(x + 1, 0) = f(x,1)$,
3. $f(x - 1, y + 1) = f(x,f(x + 1, y))$,

for all non-negative integers x,y. Determine $f(4, 1981)$

SOLUTION. Using mathematical induction, we show first that

$$f(1,y) = y + 2 \quad \text{for all } y \qquad \text{(i)}$$

When $y = 0$, $f(1,0) = f(0 + 1,0) = f(0,1)$ (by 2)
$= 1 + 1$ (by 1)
$= 0 + 2$ as asserted.

Putting $x = 0$ in 3 gives

$$f(1,y + 1) = f(0,f(1,y)) = 1 + f(1,y).$$

Hence if for some value y_0 $f(1,y_0) = y_0 + 2$ we have

$$f(1,y_0 + 1) = 1 + (y_0 + 2) = (y_0 + 1) + 2.$$

i.e. if the stated formula is true for some y, it is true for the next larger y. Hence it is true for all y.

Next we show that

$$f(2,y) = 3 + 2y \quad \text{for all } y. \qquad \text{(ii)}$$

When $y = 0$, $f(2,0) = f(1,1)$ from 2
$= 3$
$= 3 + 2 \times 0$ as asserted.

Putting $x = 1$ in 3 gives

$$f(2,y + 1) = f(1,f(2,y)) = f(2,y) + 2 \quad \text{by (i)}$$

Hence if $f(2,y) = 3 + 2y$ then $f(2,y + 1) = (3 + 2y) + 2$
$= 3 + 2(y + 1)$.

Thus (ii) has been proved by mathematical induction.

Thirdly we show that

$$f(3,y) = 2^{y+3} - 3 \qquad \text{(iii)}$$

Instead of merely proving this by induction, as for (i) and (ii) above, lets see instead how this result is arrived at.

Using 2 $f(3,0) = f(2,1) = 3 + 2$ by (ii)

Using 3 $f(3,1) = f(2,f(3,0)) = 3 + 2(3 + 2) = 3(1 + 2^1) + 2^2$

$f(3,2) = f(2,f(3,1)) = 3 + 2[3(1 + 2^1) + 2^2] = 3(1 + 2 + 2^2) + 2^3$

We can see (or guess) the repeating this process will give

$$f(3,y) = 3(1 + 2 + 2^2 + \dots + 2^y) + 2^{y+1}$$

$$= 3\left(\frac{2^{y+1} - 1}{2 - 1}\right) + 2^{y+1} = 4 \quad 2^{y+1} - 3 = 2^{y+3} - 3.$$

Now a formal proof by induction can be used to confirm the guess.

Finally we show that $f(4,y) = 2^{2^{2^{2^{\cdot^{\cdot}}}}} - 3$, where the number of 2's in the tower is $y + 3$. We revert to merely giving the inductive proof but you should have no difficulty in seeing where the result comes from, by proceeding as for (iii). When $y = 0$, $f(4,0) = f(3,1)$ (by 2) $= 2^{1+3} - 3 = 2^{2^{2}} - 3$, as asserted. Using 3

$$f(4,y+1) = f(3,f(4,y)) = 2^{f(4,y)+3} - 3$$

Hence if $f(4,y) = 2^{2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}} - 3$ where there are $(y + 3)$ 2's in the tower then

$$f(4,y+1) = 2^{(2^{2^{2^{\cdot^{\cdot^{\cdot}}}}} - 3 + 3)} - 3 = 2^{2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}} - 3$$ with one extra 2. Thus if the formula is correct for any y, it is also correct for $y + 1$. Hence it is true for all values of y.

Finally $f(4,1981) = 2^{2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}} - 3$ where there are 1984 2's in the tower. This is not a number of astronomical proportions - it is so immense that the normal run of astronomical numbers pale into microscopic insignificance!!

H.S.C. CORNER BY TREVOR will appear again in our next issue. In the last there were some misprints, did you recognise them? Please write to us if you found out! Correction will appear in the next issue.