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ROLLER COASTER DYNAMICS

BY

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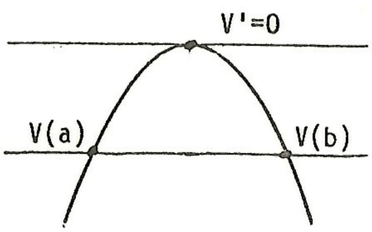
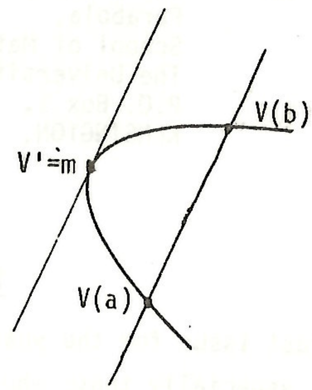
1. THE BISHOP AND ..... Perhaps the first law of motion should say "what goes up and comes down must have stopped to turn round". This would tell us to look for maxima and minima of a function $V(x)$ by considering points where $\frac{dV}{dx} = 0$. It would also tell us that between two equal values of V, say $V(a) = V(b)$, then the tangent to the curve, $y = V(x)$, must be somewhere horizontal.





Merely twisting the picture through a suitable angle shows that somewhere between a and b the gradient V' must be equal to the gradient $m = (V(b) - V(A))/(b - a)$. This, in turn, shows that any function V whose gradient V' is everywhere zero must be constant. Of course the last statement is another law of dynamics (in a sense part of Newton's first law) which says "a particle which is always instantaneously at rest never moves"! If that seems obvious - send for the bishop! The bishop in question is George Berkeley, Bishop of Cloyne (County Cork) in the 18th century who was a brilliant critic of Newton's "method of fluxions". It's not at all obvious that instantaneous velocity is well-defined, let alone that the dynamical law we have just

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stated is true. Here are some of the good bishop's words: "A method has been found to determine quantities from the velocities of their generating motion. Such velocities are called fluxions. And what are these fluxions? The velocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities nor quantities infinitely small nor yet nothing. May we not call them the ghosts of departed quantities?"

Reading through Berkeley's devastating critique of Newton's work, one cannot fail to be deeply impressed by his profound understanding of the philosophical gaps in 18th century appreciation of the fundamentals of differential calculus. On the other hand, one cannot fail to be even more deeply impressed by the fact that Newton, despite inadequate understanding, got on with the job and created beautiful and important mathematics! The moral seems clear.

2. ESCAPE FROM THE LOST PLANET. Gravitational acceleration on the surface of planet X (of radius R) is g. What is the required escape velocity for a rocketship?

The point is that we must use the inverse square law for gravitation because (hopefully!) large distances will be involved. The basic equation of motion is

$$\ddot{x} = Cx^{-2},$$

and the initial condition gives

$$-g = CR^{-2}.$$

Integrating both sides of the equation of motion with respect to x, we obtain

$$\int \ddot{x}\, dx = D - Cx^{-1} = D + gR^2x^{-1}.$$

However

$$\frac{d}{dx}(\tfrac{1}{2}\dot{x}^2) = \dot{x}\frac{d}{dx}(\dot{x}) = \dot{x}\frac{d}{dt}(\dot{x})\frac{dt}{dx} = \ddot{x}\ .$$

Therefore

$$\tfrac{1}{2}\dot{x}^2 = gR^2x^{-1} + D,$$

and consequently

$$\dot{x}^2 = 2gR^2x^{-1} + (U^2 - 2gR),$$

where U is the initial upward velocity.

Provided $U^2 > 2gR$, $\dot{x}$ always exceeds some fixed positive amount and therefore the rocketship travels on to infinity (assuming it does not fall under the gravitational influence of some other planet).

3. THE RAZOR BLADE OF LIFE. The last example admits generalization. Newton's second law of motion states that

$$\frac{d}{dt}(m\dot{x}) = F,$$

where m is mass and F the external force. In the case where m is constant and F is a function of distance x only, we have

$$m\ddot{x} = F(x).$$

Integration with respect to x (as in the previous section) gives

$$\tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 = \int_{x_1}^{x_1} F(x)dx,$$

where we wrote $v_1 = \dot{x}_1$, $v_2 = \dot{x}_2$.

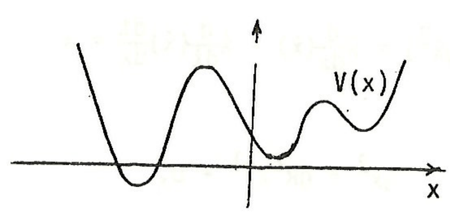
The last equation can be interpreted as saying that the change in kinetic energy equals the work done by the external force. By defining the potential energy, $V(x)$, by

$$\frac{dV}{dx} = - F(x),$$

(this defines V up to an arbitrary constant as we noted in §1) we find

$$\tfrac{1}{2}m\dot{x}^2 + V(x) = E,$$

where E is a constant.



Think of a roller-coaster travelling on the graph of $V(x)$. (Providing $V(x)$ is not too large so that we may take gravitational acceleration as constant, we may choose units so that the weight, mg, of the roller-coaster equals one unit and the height,

$V(x)$, is the potential energy in the classical sense.) Observe that points where $\frac{dV}{dx} = 0$, are equilibrium points for the traveller and that these are stable if V has a minimum and unstable if V has a maximum.

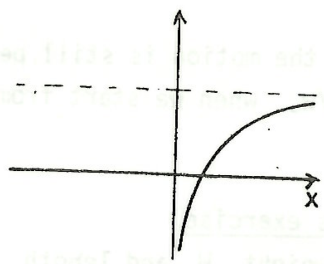
4. POTENTIAL REVEALED. Let's describe some simple examples.

a) Free fall under constant gravity

$$\ddot{x} = -g, \quad V(x) = mgx$$

b) Inverse square gravitational field

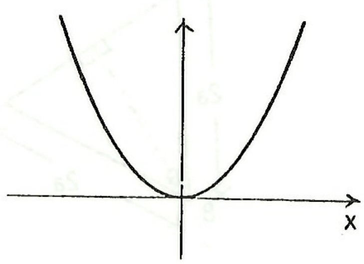
$$\ddot{x} = -g(R/x)^2, \quad V(x) = A - (B/x).$$



It is easy to see why an escape velocity should exist!

c) Harmonic oscillator

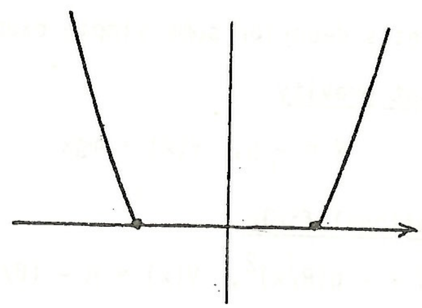
$$\ddot{x} = -w^2x, \quad V(x) = \tfrac{1}{2}w^2x^2$$



It is easy to see that we have stable oscillations about $x = 0$.

d) <u>Modified oscillator</u>

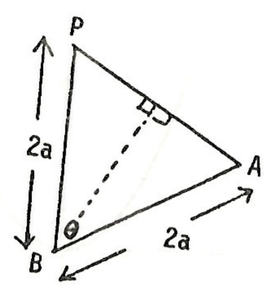
$$\left.\begin{array}{ll} \ddot{x} = -w^2 x & |x| > a, \\ \quad = 0 & |x| < a. \end{array}\right\} \qquad \left.\begin{array}{ll} V(x) = \tfrac{1}{2}w^2(x^2 - a^2) & |x| > a \\ \qquad 0 & |x| < a \end{array}\right\}$$



It is easy to see that the motion is still periodic. Try to prove that the period is $^4/_3(\pi + \sqrt{3})/w$, when we start from rest at $2a$.

e) <u>Old-fashioned mechanics exercise</u>

A uniform rod AB of weight W and length 2a is smoothly hinged at B to a fixed point. The end A is fastened to an inextensible string (of length greater than 4a) over a smooth peg at height 2a vertically above B. The other end of the string carries a weight $\tfrac{1}{2}W$. Prove that an unstable equilibrium position exists with BA making an angle of 60° to the upward vertical and a stable equilibrium position exists with A vertically below B.



The potential energy of the rod is $Wa\cos\theta$ and that of the other weight $\tfrac{1}{2}W \times PA$. However $PA = 2(2a\sin\theta/2)$, so the total potential energy is

$$V(\theta) = Wa(\cos\theta + 2\sin\theta/2).$$

$V'(\theta) = 0$ when $-\sin\theta + \cos\theta/2 = 0$ i.e. when $\cos\theta/2 = 0$ or $\sin\theta/2 = \tfrac{1}{2}$.

This gives $\theta = \pi$ or $\frac{\pi}{3}$, as required.

Also $\frac{d^2V}{d\theta^2} = Wa(-\cos\theta - \frac{1}{2}\sin\theta/2)$, so that $V'' > 0$ when $\theta = \frac{\pi}{3}$ (unstable) and $V'' < 0$ when $\theta = \pi$ (stable).

MIND BENDER!

Five students, Frank, Graham, Henrietta, Isobel and Julia were sensibly cast as the princess, the queen, a reeve, a soldier and the fairy queen Titania in their high school play. Their family names were Armstrong, Black, Carson, Davies and Elliott in some order, and the fathers' occupations were upholsterer, valuer, watchmaker, x-radiologist and yoga instructor. They each kept a pet:- kangaroo, llama, mongoose, newt or opossum.

From the following clues match up names, pets, occupations and roles.

1. Mr. Armstrong's son and the x-radiologists boy both accompanied the girl whose father was an upholsterer to visit Isobel.

2. Black escorted Henrietta to the debutant ball to see both the girl with the kangaroo and the watchmaker's daughter "come out".

3. While bushwalking with the valuer's daughter, Graham met the boy who played the reeve picnicking with Julia.

4. Neither Henrietta's nor Carson's animal can climb trees; not like the pet kept by the girl who played Titania.

5. The valuer's daughter and Titania once left rehearsal together to feed their friend's mongoose, at his special request.

6. Neither Henrietta nor Elliott took the part of the queen.

7. Julia is younger than the girl with the llama, but older than Elliott.

8. Frank's pet is not warmblooded, and his family name is not Armstrong.

(ANSWER PAGE 22.)