GEOMETRICAL TRANSFORMATIONS II

BY

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In my previous article about geometrical transformations I have described two methods, namely parallel translation and similarity transformation. The first one leaves a geometric figure unchanged but moves it parallel to itself, the second method changes the size but not the shape of the figures. Two figures related by such transformations will be said to have the same sense if, when ΔABC changes into $\Delta A'B'C'$, then both are described either clockwise or counter-clockwise. This is called direct similarity. Not every geometric transformation will preserve the "sense" of the configurations. The same way as our right hand is different from the left, an object is different from its image in a plain mirror.

Method III - Reflection. Given a straight line ℓ, let every point P be changed into its mirror image in the line ℓ. Any figure will change into a congruent figure, but described in the opposite sense called indirectly or oppositely congruent. We may combine reflection with a similarity transformation and we then consider it opposite similarity.

Problem 6. If the base and the area of a Δ are fixed, prove that the perimeter will be minimal when the Δ is isosceles.

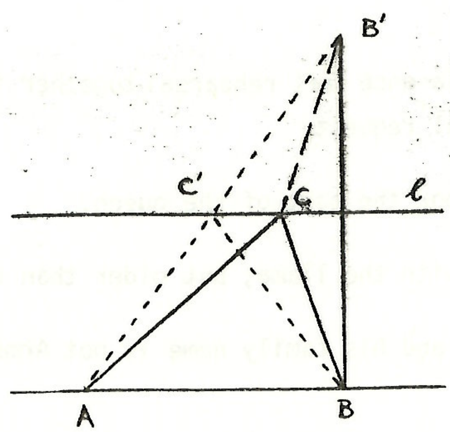


Figure 4.

Proof. (Figure 4.) As the base AB, and the area of ΔABC is constant, therefore the vertex C will describe a straight line ℓ, parallel to AB. Reflect B in ℓ getting B'. Then the perimeter of ΔABC equals

$$AB + BC + CA = BA + AC + CB'$$

this will obviously assume its minimum when

$$AC + CB' \text{ is minimal,}$$

that is when A,C,B' are 3 collinear points. Call this position of C to be C'. It is now easily proved that $\Delta AC'B$ is an isosceles Δ. (Try it!).

Method IV - Rotation. As its name suggests this method consists of rotating the unchanged configuration around a fixed point by a given angle.

Problem 7. Figure 5 is part of the well-known demonstration of Pythagaras' theorem. Triangle ABC is right-angled at C, and squares have been erected on sides AC, AB, lettered as shown. Prove that the segments BE and CG are equal in length and perpendicular to each other.

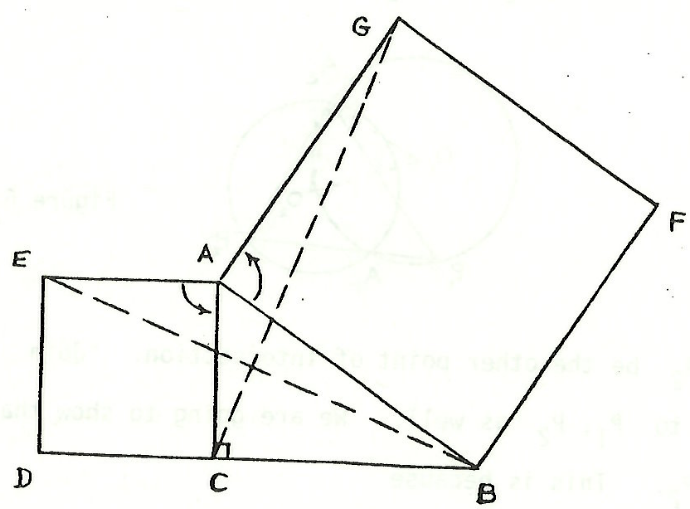


Figure 5.

Proof. Imagine that ΔAEB is rotated anti-clockwise by 90° around its vertex A. Then AB goes into AG and AE goes into AC, so the entire ΔAEB goes into ΔACG. Consequently EB must equal CG, and, together with the whole Δ, it was rotated by 90°, to cover CG.

Problem 8. If we erect a square externally on each side of an arbitrary parallelogram, the centres of these 4 squares form the vertices of a 5-th square.
(Try this problem, we will give the solution in the next issue.)

We have seen in the first article that, if two polygons are directly similar to each other and are so placed that corresponding sides are parallel to each other, then there is a central similarity between them. This means that the lines joining corresponding vertices are concurrent in a point. What can we say, if they are not situated in this fashion? The surprising answer is, that, unless one is a simple translate of the other there is in every case a "spiral similarity" between them. This is:-

Method V - Spiral similarity, meaning a rotation followed by a similarity transformation.

Problem 9. Given two intersecting circles, centres O_1, O_2, radii r_1, r_2, and a distance d. Draw a line ℓ through one of the points of intersection, A_1 to cut circle O_1 in P_1, circle O_2 in P_2, such that $\overline{P_1P_2} = d$. (See Figure 6.)

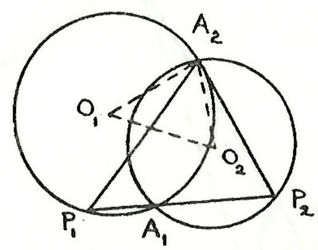


Figure 6.

Solution: Let A_2 be the other point of intersection. Join A_2 to O_1, O_2, and imagine it joined to P_1, P_2 as well. We are going to show that $\Delta O_1A_2O_2$ is similar to $\Delta P_1A_2P_2$. This is because

$$\measuredangle A_2P_1A_1 = \tfrac{1}{2}\measuredangle A_2O_1A_1 = \measuredangle A_2O_1O_2$$

and similarly

$$\measuredangle A_2P_2A_1 = \measuredangle A_2O_2O_1,$$

this means exactly that there is a spiral similarity between the $\Delta O_1A_2O_2$ and $\Delta P_1A_2P_2$. We know the lengths of O_1O_2 and P_1P_2, two corresponding sides, and this gives the ratio of all corresponding sides

$$O_1A_2 : P_1A_2 = O_1O_2 : P_1P_2,$$

so we can construct the length of A_2P_1 by the well-known method of the 4th proportional.

Now try and solve the following problem (set in the 1979-1980, Russian Olympaid).

Problem 10. Given in the plane two equilateral Δ's, ABC and A'B'C', both labelled clockwise. They are drawn so that the midpoints of the segments BC and B'C' coincide. Find

a) the angle between the lines AA' and BB' and
b) the ratio AA'/BB'.

We have now seen several methods of transformation, these are the ones most often used in solving elementary problems. There are however many other, completely different transformations. I just want to describe the most famous one, without going into details of proofs.

Inversion. Given, in the plane, a circle, centre O, radius k. Any point P in the plane will correspond to another point P', lying on the line OP such that

$$OP.OP' = k^2.$$

We can easily see some properties of this transformation. First of all the point O will have an exceptional role as it will not have an image point. Secondly, the points on the given circle, and only those, will correspond to themselves. Any point P inside the circle changes into a point outside the circle and vice versa. If P runs through a straight line passing through O, its image runs through the same line, although generally it does not coincide with P. However, if P runs through a straight line not passing through O, its image runs through a circle. If P describes a circle passing through O, its image describes a straight line. If P describes a circle not passing through O, its image describes another circle. What is important in all these changes, is that the angle between two figures remains the same, even though the shape of the figures changes. This means for instance that if a line was tangent to a circle, its image will be tangent to the image of the circle so they could be changed into two circles touching each other. Many difficult problems can be solved with the help of inversion, the most famous one historically is the problem of Apollonius:

Given 3 circles in the plane, construct a fourth circle so as to be tangent to the 3 given ones.

Solutions to problems set in article I.

Problem 2. Given two intersecting lines ℓ and m and a point P not lying on either line. Construct a straight line through P meeting ℓ in A and m in B such that P is the midpoint of AB. (Sorry about the confusing misprint in this problem.)

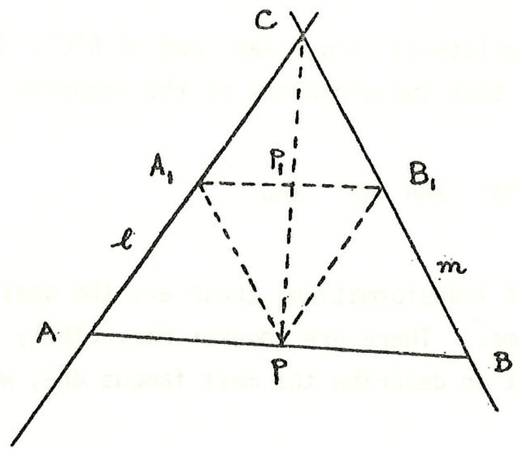


Figure 7.

Solution: (Figure 7.) Let ℓ intersect m in C. Draw a line through P parrallel to m, meeting ℓ in A_1. Similarly, let PB_1, parallel to ℓ, meet m in B_1. Let the diagonals of the parallelogram CA_1PB_1 meet in P_1. Then P_1 is the midpoint of A_1B_1 and any line segment parallel to A_1B_1, cuts ℓ and m in two points such that their midpoint is on the line CP. So the answer is draw a line through P parallel to A_1B_1.

Problem 5. Given a ΔABC, construct a square XYZU such that side XY lies along BC, while vertex Z is on AC, vertex U on AB.

Solution: (Figure 8. We construct a square BCDE, external to the triangle, on the side BC. Join D and E to the opposite vertex A, meeting BC in Y resp. X. Erect perpendiculars to BC in X and Y, meeting AB in U, AC in Z. The quadrilateral UZYX is centrally similar to the square BCDE, therefore it is itself the required square.

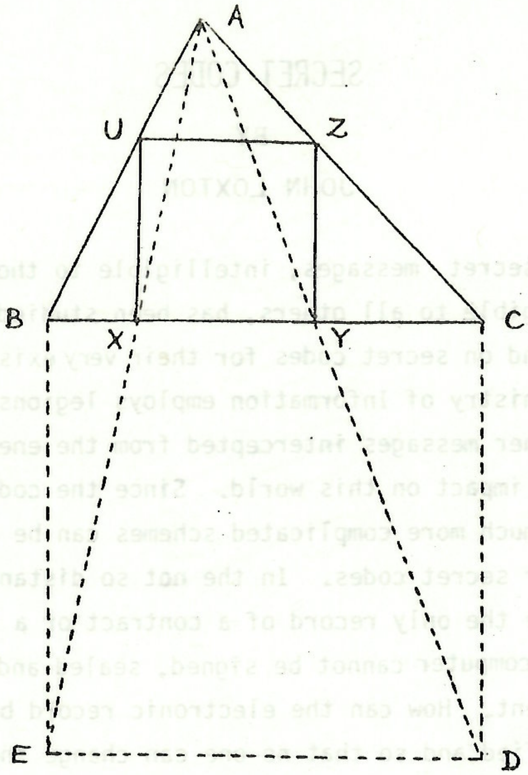
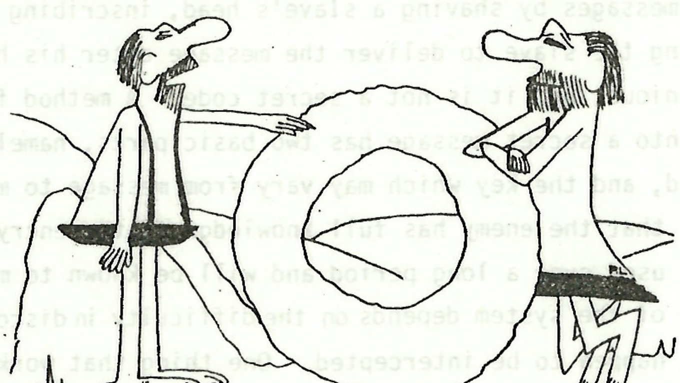


Figure 8.



WELL, I CAN SEE ITS SIGNIFICANCE AS AN ADDITIVE IDENTITY IN THE REAL NUMBER SYSTEM, BUT QUITE FRANKLY I DON'T THINK IT WILL EVER HAVE ANY PRACTICAL APPLICATION!