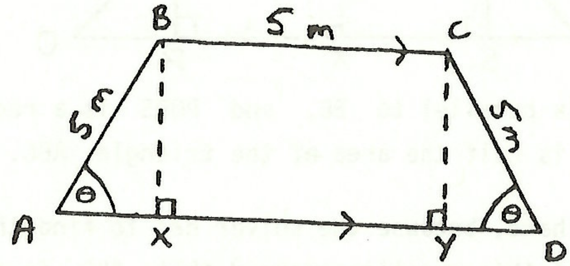
H.S.C. CORNER BY TREVOR

Some interesting applications of calculus appeared in the 1981 3 unit and 4 unit papers. Firstly two geometrical extreme value problems:

Problem 82.8. (3 unit, Q.9). In a quadrilateral ABCD, BC is parallel to AD, the sides AB, BC, CD are each 5m, and $B\hat{A}D = A\hat{D}C = \theta$. Find an expression for the area of ABCD in terms of θ, and find the value of θ for which this area is maximum.



It is surprising how many students found this area difficult to calculate. First drop the perpendiculars BX, CY from B, C to AD, as in the diagram. Clearly $BX = CY = 5\sin\theta$ and $AX = YD = 5\cos\theta$. Hence the area S is given by

$$S = 2 \times \tfrac{1}{2} \times 5\sin\theta \times 5\cos\theta + 5 \times 5\sin\theta$$
$$= 25\sin\theta(1 + \cos\theta).$$

Thus

$$\frac{dS}{d\theta} = 25[\cos\theta + \cos^2\theta - \sin^2\theta] = 25[2\cos^2\theta + \cos\theta - 1].$$

(using the fact that $\sin^2\theta = 1 - \cos^2\theta$).

$$\frac{dS}{d\theta} = 25[2\cos\theta - 1][\cos\theta + 1]$$
$$= 0 \text{ when } \cos\theta = -1, \text{ or } \cos\theta = \tfrac{1}{2}.$$

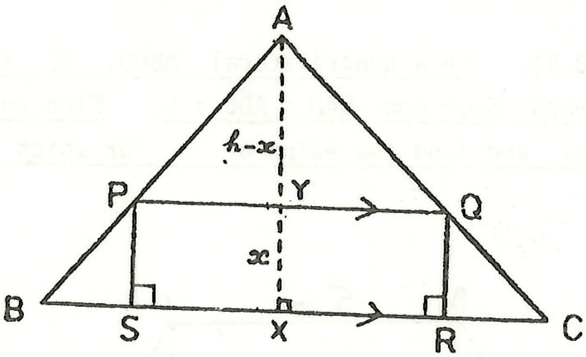
The only acute angle to satisfy this is $\theta = 60°$.

Also

$$\frac{d^2S}{d\theta^S} = 25[-\sin\theta - 4\sin\theta\cos\theta] < 0 \quad \text{when} \quad \theta = 60°.$$

Hence S is maximum when $\theta = 60°$, and then $S = \frac{75\sqrt{3}}{4}$.

Problem 82.9. (4 unit, Q. 5 (iii)).



In this figure, PQ is parallel to BC, and PQRS is a rectangle. Prove that the maximum area of PQRS is half the area of the triangle ABC.

This problem is quite hard, because the solver has to find the independent variable. Most attempts at this question assumed that ABC is equilateral, and this is neither necessary nor justified.

Let the altitude through A meet BC and PQ at X, Y, and assume $AX = h$, $BC = a$, so that area $ABC = \frac{1}{2}ah$.

Let $XY = x$, and we need to find the length PQ. Since $XY = x$, then $AY = h - x$, and, since PQ// BC, the triangles APQ, ABC, are similar. Hence $AX/AY = BC/PQ$.

Hence
$$PQ = \frac{BC.AY}{AX} \quad \frac{a(h-x)}{h}.$$

Thus area
$$PQRS = \frac{ax(h-x)}{h} = S \text{ (say)}.$$

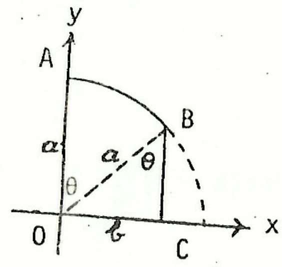
Obviously S is a quadratic expression which is maximum when $x = \frac{1}{2}h$, i.e. $\max S = \frac{1}{4}ah = \frac{1}{2}$ area ABC!

The two other parts of 4 unit Q. 5 are also quite interesting.

Problem 82.10. By reference to an appropriate diagram, or otherwise, show that for $a > b > 0$,

$$\int_0^b \sqrt{(a^2 - x^2)}dx = \tfrac{1}{2}b\sqrt{(a^2 - b^2)} + \tfrac{1}{2}a^2 \sin^{-1}\frac{b}{a}.$$

Most attempts used the substitution $x = a \sin \theta$, and this is O.K. But note $y = (a^2 - x^2)^{\frac{1}{2}}$ is the part of the circle $x^2 + y^2 = a^2$ above the x axis. Thus the integral evaluates the area enclosed by the circle, the x and y axis, and the ordinate $x = b$, as in the diagram



Required area = area of sector AOB + ΔOBC

$$= \tfrac{1}{2}a^2\theta + \tfrac{1}{2}b.BC.$$

But $\theta = \sin^{-1} b/a$, and $BC = \sqrt{(a^2 - b^2)}$, and the result follows!

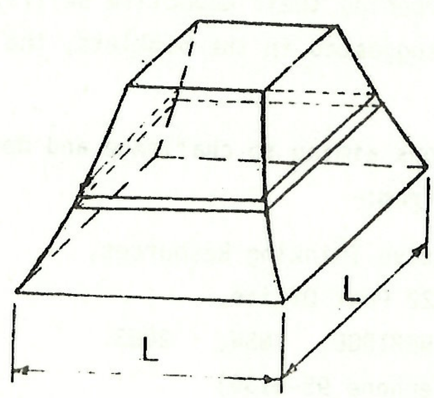
Problem 82.11. A stone building of height H metres has the shape of a flat-topped square 'pyrmid' with curved sides as shown in the figure. The cross-section at height h metres is a square with sides parallel to the sides of the base and of length

$$\ell(h) = \frac{L}{\sqrt{(h + 1)}},$$

where L is the side length of the square base in metres.

Find the volume of the building given that $H = L = 30$. Give your answer correct to the nearest cubic metre.

This is a 'slicing problem' as in the figure:



Consider a slice a height h metres from the base and of thickness dh. It is given that the slice, assumed parallel to the base, is a square of area ℓ^2, and hence

$$\ell^2 = \frac{L^2}{h+1},$$

$\therefore$ Volume of slice $= \dfrac{L^2 dh}{h+1}$.

$$\therefore \text{ Total volume } = \int_0^H \frac{L^2 dh}{h+1} = L^2[\log(h+1)]_0^H$$

$$= \underline{L^2 \log(H+1)}$$

This is an excellent illustrative example of the slicing technique.

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ANSWER TO MIND BENDER! FROM PAGE 7.

(A, G, M, S, Y); (B, F, N, R, X); (C, J, K, Q, U); (D, H, L, P, V); (E, I, O, T, W).
(The first quintuplet is to be decoded:- Armstrong, Graham, owned the mongoose, played the soldier's role, and was the son of the yoga instructor. etc.)

If you enjoy wrestling with logical puzzles such as the above, you might be interested to know that several booklets of them are available under the title MIND BENDERS, written by Anita Harnadek. The problems in booklets A1 to A4 are easy, those in B1 to B4 are of medium difficulty, and those in C1 to C3 are more difficult. I would suppose that many Parabola readers would find the B and C level books a ton of fun, as well as an aid for sharpening their deductive skills. Even using the systematic method of solution suggested in the booklets, the C level problems do not yield without a struggle

These and other publications aiming to challenge and develop various areas of mental ability are obtainable from:-

Creative Thinking Resources,
Box 22 Post Office,
NORTHBRIDGE, NSW, 2063.
(Telephone 95-5136)

Answers to secret codes from page 18.

(1) To be, or not to be: that is the question:
Whether 'tis nobler in the mind to suffer
The slings and arrows of outrageous fortune,
Or to take up arms against a sea of troubles,
And by opposing end them?

(2) The content and depth of treatment of this course as specified in the syllabus and notes indicate that it is intended for students who have completed the School Certificate mathematics course.

(3) Theorem. The square on the hypotenuse is equal to the sum of the squares on the other two sides.

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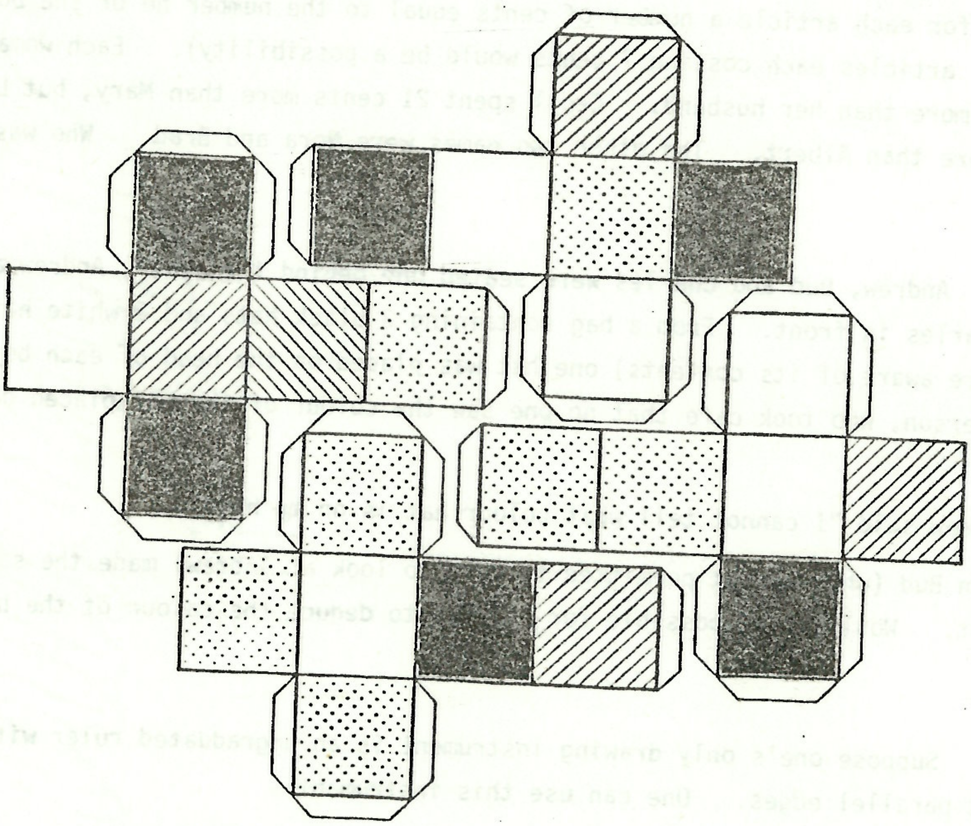

Above are the plans of four cubes with faces coloured by 4 colours as shown. Place them on top of each other to form a column (parallelepiped) so that the four colours appear on each vertical face.