PROBLEMS

Problem Q. 539 has been suggested by a reader who would like help in solving it. Problems Q. 546 - Q. 550 are five of the six questions faced by 119 students from 30 different countries at this years International Mathematical Olympiad. Four Australian students went to Budapest in July to take part. Time allowed - 2 sessions of 4½ hours, 3 questions being presented at each session. While 3 astonishing students succeeded in finding faultless answers to all 6 questions, you don't have to feel too humiliated if you can't do any of them. At least a dozen of the students in Budapest didn't get close to a complete solution of a single problem.

Q. 539. From the set of whole numbers {0, 1, 2, ..., 999999999} two are selected at random. What is the probability that they differ by a multiple of 10000?

Q. 540. Three married couples went shopping. Each of the six bought several articles and paid for each article a number of cents equal to the number he or she bought. (e.g. 7 articles each costing 7 cents would be a possibility). Each woman spent 75 cents more than her husband. Cecil spent 21 cents more than Mary, but Leah spent $14.19 more than Albert. The other two names were Nora and Brad. Who was married to whom?

Q. 541. Andrew, Bud and Charles were seated one behind the other, Andrew at the back, Charles in front. From a bag containing 3 black hats and 2 white hats (all three were aware of its contents) one hat was placed on the head of each boy by a fourth person, who took care that no one saw the colour of the hat placed on his own head.

Andrew said "I cannot tell what colour hat is on my head".

Then Bud (who was not permitted to turn to look at Andrew) made the same statement. Would it be possible for Charles to deduce the colour of the hat on his head?

Q. 542. Suppose one's only drawing instrument is an ungraduated ruler with two straight parallel edges. One can use this instrument:

a) to draw a line through two given (or already constructed) points.

b) to draw a line parallel to a given line at a distance from it equal to the width of the ruler.

c) to draw two parallel lines, one through each of two given points whose distance apart is at least equal to the width of the ruler.

Show how to use this instrument to

1. trisect a line segment of length greater than the width of the ruler.
2. trisect a line segment of length less than the width of the ruler.

Q. 543. Six different lengths are given, such that they can be taken in any order as the edges of a tetrahedron ABCD.

How many non-congruent tetrahedra can be constructed in this way? (Count "mirror image" tetrahedra as non-congruent.)

Q. 544. (i) Given an 8×8 chessboard, what is the largest number of bishops which can be placed on the board so that no bishop attacks any other?

(ii) In how many different ways can this maximum number of bishops be placed on the board? (A bishop moves diagonally in any direction, any number of squares.)

Q. 545. The squares of an $n \times n$ chessboard are to be coloured red, yellow, blue or green in such a way that squares with a common edge or corner are given different colours. Is it possible to do this in such a way that every row or column contains at least one square of each colour?

Q. 546. The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m,n,

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1;$$

$$f(2) = 0,\ f(3) > 0, \text{ and } f(9999) = 3333.$$

Determine $f(1982)$.

Q. 547. Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1 \text{ and for all } i \geq 0,\ x_{i+1} \leq x_i.$$

a) Prove that for every such sequence, there is an $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \dots + \frac{x_{n-1}^2}{x_n} \geq 3.999 .$$

b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \ldots + \frac{x_{n-1}^2}{x_n} < 4 \quad \text{for all} \quad n.$$

Q. 548. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x,y), then it has at least three such solutions. Show that the equation has no solution in integers when $n = 2891$.

Q. 549. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M, and N are collinear.

Q. 550. Let S be a square with sides of length 100 and let L be a path within S which does not meet itself and which is composed of linear segments $A_0A_1, A_1A_2, \ldots, A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than 1/2. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1 and the length of that part of L which lies between X and Y is not smaller than 198.

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PROBLEM SOLVERS FOR THE PROBLEMS IN VOLUME 18, NUMBER 1.

The following students submitted solutions to the questions indicated.

Rolfe Bozier (Barker College) - Q516, Q517, Q521, Q525.

Allan Mekisic (North Sydney Boys' High School) - Q517.

Janine Mok (Pymble Ladies College) - Q515.

A. Robson (Barker College) Q517.