SOLUTIONS TO PROBLEMS FROM VOLUME 18, NUMBER 1

Q. 515. I have two different integers > 1. I inform Sam and Pam of this fact and I tell Sam the sum of my two numbers and I tell Pam their product. The following dialogue then occurs:

Pam: I can't determine the numbers.
Sam: The sum is less than 23.
Pam: Now I know the numbers.
Sam: Now I know the numbers too.

What are the numbers?

SOLUTION: The product P, of the two numbers must have at least two different non-trivial factorisations $P = x \cdot y$, $x \neq y$, $2 \leqslant x$, $2 \leqslant y$ for exactly one of which the sum of the factors is less than 23. Numbers less than 2×21 have no factorisations with the sum exceeding 22, and numbers greater than 121 have no factorisations with the sum less than 23. The possible values of P satisfying these conditions can be found with the expenditure of a little effort, and prove to be the following:-

120, 117, 105;
110, 108, 104, 98, 68;
99, 75, 64;
88, 78;
66, 52;
63;
50, 44.

They have been placed in groups as follows:- If the only allowable factorisation of each of the numbers in any one group is found, the sum of the factors is the same for all members of the group. For example, in the first group, the only permissible factorisations are 10×12; 9×13; and 7×15. In each case, the sum of the factors is 22.

The sum of factors in the other groups is 21, 20, 19, 17, 16 and 15 respectively.

Which of these sums was given to Sam?

Could it have been 22?

After Pam's declaration that she knew the numbers, Sam would have been able to conclude that the product known to her was one of 120, 117 and 105; but he would not have been able to decide which. Well, it is now clear that the only possible value of the sum given to Sam is 16, since this corresponds to the only group of products containing just one entry, viz, 63.

Thus the two numbers were 7 and 9.

Q. 516. Solve the following equation:

$$x = 1 + \cfrac{2}{1 + \cfrac{2}{1 + \cfrac{\ddots}{1 + \cfrac{2}{1 + \frac{2}{x}}}}}$$

(There are n vinculi (division lines) in the expression on the RHS.)

SOLUTION: The RHS simplifies to an expression of the form $\frac{Ax + B}{Cx + D}$ where A,B,C,D are positive numbers (except that $D = 0$ if $n = 1$). This can readily be proved by induction: Assume the above result when there are n vinculi, and consider a similar expression with $(n + 1)$ vinculi. Replace the final $1 + \frac{2}{x}$ by y. By assumption (since only n vinculi remain) the expression is equal to

$$\frac{Ay + B}{Cy + D} = \frac{A(1 + \frac{2}{x}) + B}{C(1 + \frac{2}{x}) + D} = \frac{(A + B)x + 2A}{(C + D)x + 2C}$$

which is still of the desired form.

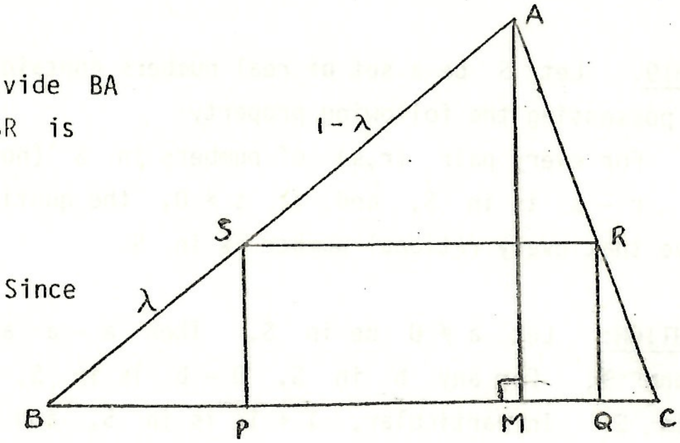
It follows that the given equation reduces to

$$x = \frac{Ax + B}{Cx + D} \text{ or } Cx^2 + (D - A)x - B = 0 ,$$

a quadratic equation; which can have no more than two roots.

But it is trivial to check that $x = 2$ and $x = - 1$ satisfy the given equation no matter how many division lines are on the RHS, since for these numbers $1 + \frac{2}{x} = x$. Hence they are the only solutions for the given equation.

Q. 517. Let P,Q,R,S be four points on the sides of a triangle ABC such that PQRS is a rectangle. Show that the maximum area of the rectangle PQRS is half the area of the triangle ABC.



SOLUTION: In the figure, let S divide BA in the ratio $\lambda : 1 - \lambda$. Since ΔASR is similar to ΔABC

$$\frac{SR}{BC} = \frac{AS}{AB} = \frac{1-\lambda}{1}$$

Let AM be perpendicular to BC. Since ΔBSP is similar to ΔBAM

$$\frac{PS}{MA} = \frac{BS}{BA} = \frac{\lambda}{1}$$

$\therefore$ Area of rectangle $= SR.PS$

$= (1 - \lambda)BC.\lambda AM$

$= 2\lambda(1 - \lambda)$.area of ΔABC

Now $\lambda(1 - \lambda) = \frac{1}{4} - (\frac{1}{4} - \lambda + \lambda^2) = \frac{1}{4} - (\lambda - \frac{1}{2})^2$

$\leqslant \frac{1}{4}$ for all λ since squares are never negative.

$\therefore$ Area of PQRS $\leqslant 2.\frac{1}{4}$ area of $\Delta ABC = \frac{1}{2}$ area of ΔABC. Equality is achieved by taking $\lambda = \frac{1}{2}$.

Q. 518. Find a polynomial with integer coefficients having $\sqrt{2} + \sqrt{3}$ and $\sqrt{2} + \sqrt[3]{3}$ as roots.

SOLUTION: Let $f(x)$ be a polynomial with a root α, and $g(x)$ a polynomial with a root β. Then if $h(x) = f(x)g(x)$, $h(x)$ vanishes for $x = \alpha$ and for $x = \beta$. Let $\alpha = \sqrt{2} + \sqrt{3}$. Then $(\alpha - \sqrt{2})^2 = 3$, from which $\alpha^2 - 1 = 2\sqrt{2}\,\alpha$. Squaring yields $\alpha^4 - 2\alpha^2 + 1 = 8\alpha^2$, so that α is a root of $f(x) = x^4 - 10x^2 + 1$.

Let $\beta = \sqrt{2} + \sqrt[3]{3}$. Then $(\beta - \sqrt{2})^3 = 3$, from which

$$\beta^3 + 6\beta - 3 = \sqrt{2}(3\beta^2 + 2)$$

Squaring yields

$$\beta^6 + 12\beta^4 - 6\beta^3 + 36\beta^2 - 36\beta + 9 = 2(9\beta^4 + 12\beta^2 + 4)$$

so that β is a root of

$$g(x) = x^6 - 6x^4 - 6x^3 + 12x^2 - 36x + 1.$$

An answer to the problem is thus given by

$$f(x)g(x) = x^{10} - 16x^8 - 6x^7 + 73x^6 + 24x^5 - 125x^4 + 354x^3 + 2x^2 - 36x + 1.$$

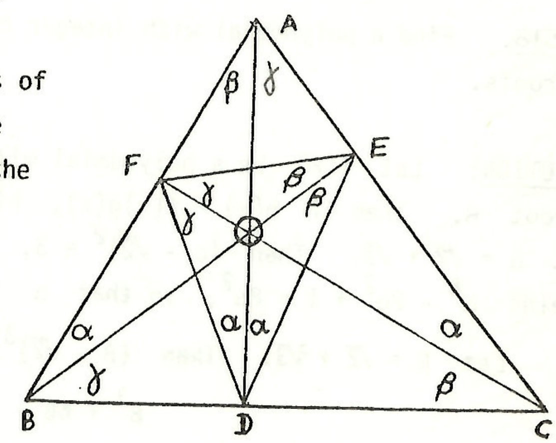
Q. 519. Let S be a set of real numbers containing at least one non-zero number, and possessing the following property:

For every pair (r,s) of numbers in S (not necessarily different), $r - s$ is in S, and, if $s \neq 0$, the quotient r/s is in S.

Prove that every rational number is in S.

SOLUTION: Let $a \neq 0$ be in S. Then $a - a$ and a/a are in S, i.e., S contains 0 and 1. For any b in S, $0 - b$ is in S, and therefore $a - (-b) = a + b$ is in S. In particular, $1 + 1$ is in S, and then $1 + 1 + 1$ is in S etc; i.e. all the positive integers are in S and also all the negative integers. Since every rational number is expressible as the quotient of integers with the denominator not zero, the desired result has been established.

Q. 520. Let DEF be the pedal triangle of the acute angled triangle ABC (i.e., D,E,F are the feet of the perpendiculars from A,B,C to BC, CA, AB respectively). Show that the greatest angle of triangle DEF is at least as large as the greatest angle of ABC. When does one have equality?



SOLUTION: If necessary relabel the vertices of the triangle so that $\angle A \geq \angle B \geq \angle C$. In the figure AD, BE and CF are the altitudes of the triangle ABC. $\angle ABE = 90 - \angle A$ from the right angled triangle ΔAEB. This angle is equal to $\angle FDO$ since they both stand on the chord FO in the cyclic quadrilateral $BDOF$. Similarly $\angle ODE = \angle OCE = 90 - \angle A$. In the same way, the four angles marked β in the figure are all equal to $90 - \angle B$, and those marked γ are $90 - \angle C$.

Since $90 - \angle A \leq 90 - \angle B \leq 90 - \angle C$ we have $\alpha \leq \beta \leq \gamma$. The largest angle of the pedal triangle is therefore $\angle DFE = 2\gamma \geq \beta + \gamma = \angle A$ = largest angle of $\angle ABC$.

Here we have equality if $\gamma = \beta$, i.e. if $\angle B = 90 - \beta = 90 - \gamma = \angle C$. Thus the largest angle of the pedal triangle is the same size as the largest angle of the given triangle if and only if the smallest two angles of ΔABC are equal; i.e. if ΔABC is isosceles with the equal sides forming an angle of at least $60°$.

Q. 521. Find all pairs of integers x,y such that

$$x^3 - y^3 = 1729$$

Prove that there are no others.

SOLUTION: To solve $x^3 - y^3 = (x - y)(x^2 + xy + y^2) = 1729$ in integers, first note that $x^2 + xy + y^2 = (x + \frac{1}{2}y)^2 + \frac{3}{4}y^2$ is never negative, so that $x - y$ and $x^2 + xy + y^2$ must both be positive integer factors of 1729, the first not larger than the second. (For integers (x,y), $x^2 + xy + y^2 \geqslant x - y$ except when $(x,y) = (1, -1)$.)

Since $1729 = 7 \times 13 \times 19$ the problem reduces to solving the simultaneous equations

$$x - y = \alpha \qquad (1)$$

$$x^2 + xy + y^2 = \beta \qquad (2)$$

in integers (x,y), where $(\alpha,\beta) = (1, 1729)$, or $(7, 247)$, or $(13, 133)$, or $(19, 91)$.

Eliminating x between the two equations gives $3y^2 + 3\alpha y + (\alpha^2 - \beta) = 0$. for the four sets of values of (α,β) this becomes

$$y^2 + y - 576 = 0$$

or

$$y^2 + 7y - 66 = 0$$

or

$$y^2 + 13y + 12 = 0$$

or

$$y^2 + 19y + 90 = 0.$$

The first two of these have no integer solutions. The third has solutions $y = -1$, or -12; the corresponding values of $x (= y + \alpha = y + 13)$ are $+12$ and $+1$ respectively.

The fourth equation has solutions $y = -9$ or -10, the corresponding values of x being $+10$, and $+9$ respectively.

Hence there are exactly 4 solutions in integers of the given equation; viz

$$(x,y) = (12, -1), \text{ or } (1, -12), \text{ or } (10, -9), \text{ or } (9, -10).$$

Q. 522. Show that there exist two constants A and B $(A > B)$ such that, if f is the function defined for all $x \neq -1/B$ by

$$f(x) = (1 + Ax)/(1 + Bx),$$

then the expression $f(1/(1 + 2x))/f(x)$ is independent of x (i.e., constant wherever it is defined). Next, consider the list $(a(n))$ defined by

$$a(0) = 1, \quad a(n + 1) = 1/(1 + 2a(n)) \quad \text{for } n = 0,1,2, \ldots$$

By considering also the list $f(a(0))$, $f(a(1))$, $f(a(2))$, ... find a formula giving the value of $a(n)$ for every n.

SOLUTION:

$$f\left(\frac{1}{(1 + 2x)}\right)/f(x) \qquad (1)$$

$$= \frac{1 + \frac{A}{1 + 2x}}{1 + \frac{B}{1 + 2x}} \cdot \frac{1 + Bx}{1 + Ax}$$

$$= \frac{((A + 1) + 2x)}{((B + 1) + 2x)} \frac{(1 + Bx)}{(1 + Ax)}$$

This yields a constant function if the polynomial $1 + Ax$ divides exactly into $(A + 1) + 2x$ and $1 + Bx$ divides exactly into $(B + 1) + 2x$; i.e. if $\frac{A + 1}{1} = \frac{2}{A}$ and $\frac{B + 1}{1} = \frac{2}{B}$. Thus the expression (1) is independent of x if A and B are both roots of $y^2 + y - 2 = 0$. Since $A > B$ this gives $A = 1$, $B = -2$.

Indeed, for these values of A and B, it can be confirmed that $f(\frac{1}{1 + 2x})/f(x)$ has the constant value -2. (2)

Now consider the list $f(a(0))$, $f(a(1))$, $f(a(2))$, ...

Since

$$a(n + 1) = \frac{1}{1 + 2a(n)}$$

$$f(a(n + 1)) = f\left(\frac{1}{1 + 2a(n)}\right) = (-2).f(a(n)) \quad \text{by (2).}$$

Now

$$f(a(0)) = f(1) = \frac{1 + A.1}{1 + B.1} = \frac{1 + 1}{1 - 2} = -2,$$

so that

$$f(a(1)) = (-2).f(a(0)) = (-2)^2,$$

$$f(a(2)) = (-2).f(a(1)) = (-2)^3, \text{ etc.}$$

It is clear that, for all n,

$$f(a_n) = (-2)^{n+1}$$

i.e. $$\frac{1 + 1.a_n}{1 - 2.a_n} = (-2)^{n+1}$$

Thus $$((-2)^{n+2} - 1)\, a_n = -(-2)^{n+1} + 1$$

or $$a_n = \frac{1 - (-2)^{n+1}}{-1 + (-2)^{n+2}}$$

for every n.

Q. 523. Let S be the set of polynomials of the form

$$f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

such that $|f(x)| \leqslant 1$ whenever $|x| \leqslant 1$. Show that there exists a number M such that $|a_3| \leqslant M$ for all polynomials in S. Try to find a value for M as small as possible.

SOLUTION: If $-1 \leqslant a_3x^3 + a_2x^2 + a_1x + a_0 \leqslant 1$ for all x in $[-1, 1]$ then, since $-x$ is still in $[-1, 1]$

$$-1 \leqslant -[a_3(-x)^3 + a_2(-x)^2 + a_1(-x) + a_0] \leqslant 1.$$

Adding these inequalities and dividing by 2 yields

$$-1 \leqslant a_3x^3 + a_1x \leqslant 1 \quad \text{for all } x \text{ in } [-1, 1]$$

In particular, taking $x = 1$ and $x = \frac{1}{2}$, we have

$$-1 \leqslant a_3 + a_1 \leqslant 1 \qquad (1)$$

and $$-1 \leqslant \frac{a_3}{8} + \frac{a_1}{2} \leqslant 1 \qquad (2)$$

Multiplying (2) by -2 gives

$$-2 \leqslant -\frac{a_3}{4} - a_1 \leqslant 2 \qquad (3)$$

which, added to (1) yields

$$-3 \leqslant \frac{3a_3}{4} \leqslant +3.$$

and $$-4 \leqslant a_3 \leqslant 4$$

Thus for all polynomials in S it is true that $|a_3| \leqslant 4$.

As it happens, $M = 4$ is the best possible value, as may be shown by exhibiting a polynomial in S with a_3 equal to 4. Such a polynomial is $4x^3 - 3x$, whose value is in $[-1, 1]$ for all x in $[-1, 1]$, as is evident from the trigonometric identity

$$4 \cos^3 \theta - 3 \cos \theta = \cos 3\theta.$$

<u>Q. 524</u>. Show that, for every positive integer n,

$$2(\sqrt{(n+1)} - \sqrt{n}) < 1/\sqrt{n} < 2(\sqrt{n} - \sqrt{(n-1)}).$$

Find the largest integer less than

$$1 + 1/\sqrt{2} + 1/\sqrt{3} + \ldots + 1/\sqrt{10000}.$$

<u>SOLUTION</u>: $\sqrt{n+1} > \sqrt{n} > \sqrt{n-1}$, for every positive integer n;

$$\Rightarrow \quad \sqrt{n+1} + \sqrt{n} > 2\sqrt{n} > \sqrt{n} + \sqrt{n-1}$$

$$\Rightarrow \quad \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n} + \sqrt{n-1}}$$

$$\Rightarrow \quad \frac{\sqrt{n+1} - \sqrt{n}}{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})} < \frac{1}{2\sqrt{n}} < \frac{\sqrt{n} - \sqrt{n-1}}{(\sqrt{n} + \sqrt{n-1})(\sqrt{n} - \sqrt{n-1})}$$

$$\Rightarrow \quad 2(\sqrt{n+1} - \sqrt{n}) < \frac{1}{\sqrt{n}} < 2(\sqrt{n} - \sqrt{n-1}).$$

Thus

$$2(\sqrt{6} - \sqrt{5}) < \frac{1}{\sqrt{5}} < 2(\sqrt{5} - \sqrt{4})$$

$$2(\sqrt{7} - \sqrt{6}) < \frac{1}{\sqrt{6}} < 2(\sqrt{6} - \sqrt{5})$$

$$\vdots$$

$$2(\sqrt{10001} \quad \sqrt{10000}) < \frac{1}{\sqrt{10000}} < 2(\sqrt{10000} - \sqrt{9999}).$$

Adding, we obtain

$$2(\sqrt{10001} - \sqrt{5}) < \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} + \ldots + \frac{1}{\sqrt{10000}} < 2(\sqrt{10000} - \sqrt{4}).$$

Since $\sqrt{10001} \approx 100.005$ and $\sqrt{5} \approx 2.23607$ we obtain

$$195.538 < \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} + \ldots + \frac{1}{\sqrt{1000}} < 196$$

Adding $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} \approx 1 + .707 + .577 + .5 = 2.784$ to all terms gives

$$198.3 < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{10000}} < 198.784.$$

Hence the required integer is 198.

Q. 525. A point in the Cartesian plane is called an integer point if its coordinates (x,y), are both integers. Denote by $A(R)$ the number of integer points lying inside the disc of radius R, centre the origin.

a) Show that the expression $A(R)/R^2$ approaches a limiting value, ℓ, when R tends to infinity. Find ℓ.

b) For all R, let

$$B(R) = A(R) - \ell R^2$$

Then for $k = 2$, it is certainly true that $B(R)/R^k$ approaches 0 as R tends to infinity. For which values of k less than 2 is this statement still true?

SOLUTION: Let $N_1(R)$ denote the number of integer points (x,y) with x and y both positive lying inside the disc of radius R, and $N_2(R)$ the number of integer points $(x,0)$ inside the disc with $x > 0$. Then $A(R) = 4N_1(R) + 4N_2(R) + 1$.

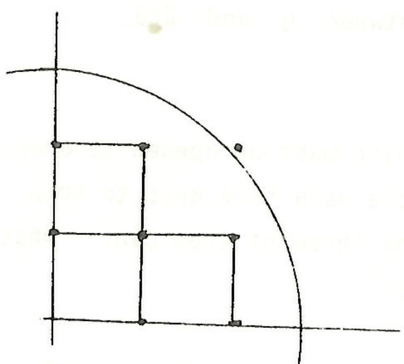


Figure 1.

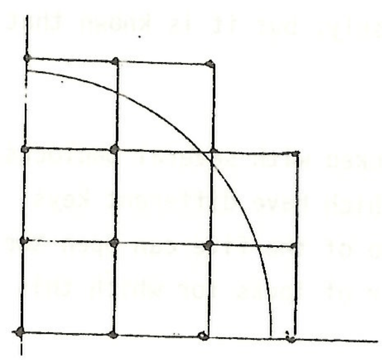


Figure 2.

In Figure 1, unit squares have been drawn having as the upper right corner an integer point in the first quadrant of the circle (R a little greater than $\sqrt{5}$ in the diagram). It is clear that the area of the quadrant exceeds the total area of all the squares

$$N_1(R) < \tfrac{1}{4}\pi R^2$$

$$A(R) < \pi R^2 + 4N_2(R) + 1$$

$$< \pi R^2 + 4R + 1 \qquad (\text{since } N_2(R) < R)$$

In Figure 2, unit squares have been drawn having as the lower left corner an integer point inside the first quadrant of the circle or on the axes bordering it. We see that

$$\tfrac{1}{4}\pi R^2 < N_1(R) + 2N_2(R) + 1$$

From this $$\pi R^2 - 4N_2(R) - 3 < A(R).$$

Thus $$\pi R^2 - 4R - 3 < A(R) < \pi R^2 + 4R + 1 \quad \text{for all } R.$$

It is now clear that $\frac{A(R)}{R^2}$ lies between $\pi - \frac{4}{R} - \frac{3}{R^2}$ and $\pi + \frac{4}{R} + \frac{1}{R^2}$ for all R, whence it approaches the limiting value π as $R \to \infty$.

Now if $B(R) = A(R) - \pi R^2$ we have

$$-4R - 3 < B(R) < 4R + 1 \quad \text{for all } R$$

and it is immediately clear that $\frac{B(R)}{R^k}$ approaches 0 if k is any number greater then 1.

Are there values of $k \leqslant 1$ for which $\frac{B(R)}{R^k}$ still tends to 0? The answer is yes, but more difficult methods are needed for the proofs. The greatest lower bound for k (roughly speaking, the smallest value of k such that $\frac{B(R)}{R^k}$ tends to 0) is not known precisely, but it is known that it lies between ½ and 2/3.

Q. 526. A box is locked with several padlocks, all of which must be opened to open the box, and all of which have different keys. Five people each have keys to some of the locks. No two of the five can open the box but any three of them can. What is the smallest number of locks for which this is possible?

SOLUTION: Let the five people be denoted by A, B, C, D and E and write on each lock the letters of people who do not hold a key for that lock. Then no lock has three or more letters on it (since any three people can open the box). For any two people, there is at least one lock with the corresponding two letters. Since there are ${}^5C_2 = 10$ different pairs of letters, there must be at least 10 locks, bearing the inscriptions AB, AC, AD, AE, BC, BD, BE, CD, CE, DE (Note that each of the five people holds 6 keys.)