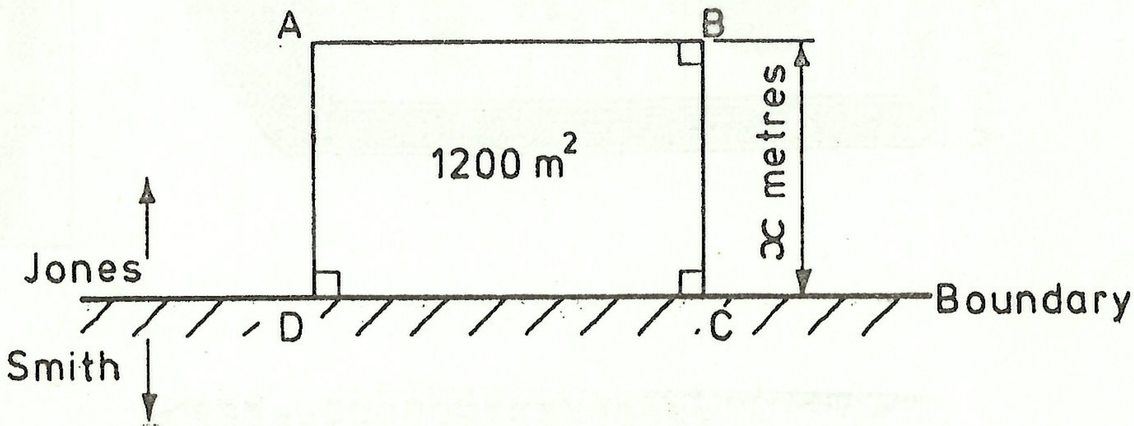
H.S.C. CORNER BY TREVOR

There were once again some interesting problems in the 1982 H.S.C. Turning first to some extreme value problems, starting with the 3 unit paper:

Problem 83.1. Farmer Jones has to build a fence to enclose a 1200 m^2 area ABCD as in the diagram. Fencing costs \$3 per metre, but Farmer Smith has agreed to pay half the cost of fencing CD. What is the maximum amount Jones has to pay?



This is a variant of a very well-known optimisation problem, and is solved as follows:

Since the area of ABCD is 1200 m^2, AB = CD = 1200/x m. Thus the cost \$y of fencing is given by

$$y = 3(AB + BC + AD) + \frac{3}{2}DC$$

$$= 6x + 5400/x$$

$$\therefore \quad \frac{dy}{dx} = 6 - 5400/x^2; \quad \text{and} \quad \frac{d^2y}{dx^2} = 10800/x^3 > 0.$$

Thus y is minimum when $\frac{dy}{dx} = 0$, and then $x^2 = 900$, or $x = 30$ m. The minimum cost to Jones is \$360.

Problem 83.2: On the 4 unit paper, P is given to be a point on the curve $x^4 + y^4 = 1$, and it is required to prove that the distance of P from the origin is at most $2^{\frac{1}{4}}$.

This problem can be bludgeoned out using calculus, since, again $y^4 = 1 - x^4$, then the distance z of P from 0 is given by

$$w = z^2 = x^2 + (1 - x^4)^{\frac{1}{2}}$$

so that

$$\frac{dw}{dx} = 2x - \frac{2x^3}{(1 - x^4)^{\frac{1}{2}}} ; \quad \frac{d^2w}{dx^2} = 2 - \frac{6x^2}{(1 - x^4)^{\frac{1}{2}}} + \frac{4x^6}{(1 - x^4)^{3/2}} .$$

Now $\frac{dw}{dz} = 0$ when $x = 0$, or $x^2 = (1 - x^4)^{\frac{1}{2}}$, i.e. $2x^4 = 1$, so that the turning points of w are at $x = 0$, and $x = 2^{-\frac{1}{4}}$. When $x = 0$, $\frac{d^2w}{dz^2} > 0$, and when $x = 2^{-\frac{1}{4}}$, $\frac{d^2w}{dz^2} < 0$, and hence w is minimum when $x = 0$, and maximum when $x = 2^{-\frac{1}{4}}$, and $w = 1, 2^{\frac{1}{2}}$ respectively. Since $w = z^2$, we find that $1 \leqslant z \leqslant 2^{\frac{1}{4}}$, and the result is proved.

There are two other neater, trickier ways, however.

(i) Note that $(x^2 - y^2)^2 \geqslant 0$, and hence $x^4 + y^4 \geqslant 2x^2y^2$. Hence $2x^2y^2 \leqslant 1$. But let $z^2 = x^2 + y^2$, then $z^4 = x^4 + y^4 + 2x^2y^2 = 1 + 2x^2y^2 \leqslant 2$. Thus $z \leqslant 2^{\frac{1}{4}}$.

(ii) Let $x = r\cos\theta$, $y = r\sin\theta$, and hence $z^2 = x^2 + y^2 = r^2$. But

$$x^4 + y^4 = r^4(\cos^4\theta + \sin^4\theta) = r^4[(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cos^2\theta]$$

$$\therefore \quad 1 = r^4(1 - \tfrac{1}{2}\sin^2 2\theta)$$

$$\therefore \quad r^4 = \{1 - \tfrac{1}{2}\sin^2 2\theta\}^{-1}$$

But $$0 \leqslant \sin 2\theta \leqslant 1.$$

$$\therefore \quad 1 \geqslant 1 - \tfrac{1}{2}\sin^2 2\theta \geqslant \tfrac{1}{2}$$

$$\therefore \quad 1 \leqslant r^4 \leqslant 2$$

and the result follows!

Problem 83.3. Given that $a_n = \surd(2 + a_{n-1})$ for integers $n \geqslant 1$, and $a_0 = 1$, prove that, for $n \geqslant 1$, $\surd 2 < a_n < 2$.

Proof is by induction. Assume that

$$\sqrt{2} < a_n < 2.$$

Now $a_{n+1} = \surd(2 + a_n)$. Since $a_n > \sqrt{2}$, then $a_{n+1} > \surd(2 + \sqrt{2}) > \sqrt{2}$. Also, when $a_n < 2$, $a_{n+1} < \surd(2 + 2) = 2$. Thus $\sqrt{2} < a_{n+1} < 2$.

But $a_1 = \sqrt{3}$, and hence $\sqrt{2} < a_1 < 2$. Thus we have proved that if the result is true for n, then it is also true for $n + 1$. It is true for $n = 1$, and hence, by induction, for all $n \geqslant 1$.

This is a very simple application of proof by induction, but surprisingly few correct answers were received. Perhaps the slightly unfamiliar algebra tripped up a lot of students!

We finish this month's article by considering a trigonometric equation.

Problem 83.4. Find all x such that $\cos x + \sin x = 1 + \sin 2x$, and $0 \leqslant x \leqslant 2\pi$.

Solution:

$$\begin{aligned} \cos x + \sin x &= 1 + 2 \sin x \cos x \\ &= \cos^2 x + \sin^2 x + 2 \sin x \cos x \\ &= (\cos x + \sin x)^2. \end{aligned}$$

Now when $u = u^2$, it follows that $u - u^2 = 0$, i.e. $u(1 - u) = 0$, so that $u = 0,1$. Thus either $\cos x + \sin x = 0$, or $\cos x + \sin x = 1$. [Many students missed one or the other of these solutions!]

a) When $\cos x + \sin x = 0$, then $1 + \tan x = 0$, i.e. $\tan x = -1$, and $x = \frac{3\pi}{4} + n\pi$, $n = 0, \pm 1, \pm 2$ etc. This yields two solutions, $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, in the required range.

b) When

$$\cos x + \sin x = 1.$$

Note that $\cos x + \sin x = \sqrt{2} \sin(x + \frac{\pi}{4})$

$$\sin(x + \tfrac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

$$\therefore \qquad x + \frac{\pi}{4} = \frac{\pi}{4} + 2n\pi, \quad \text{or} \quad \frac{3\pi}{4} + 2n\pi$$

$$\therefore \qquad x = 0,\ 2\pi, \ \text{and} \ \frac{\pi}{2}$$

Thus the solutions in the required range are $\boxed{0, \frac{\pi}{2}, 2\pi, \frac{3\pi}{4}, \frac{7\pi}{4}}$.

Some candidates tried the formula $t = \tan\frac{x}{2}$. This leads to the correct solution after some very difficult algebra (try it for yourself). Quite a lot of students tried squaring both sides as follows

$$(\sin x + \cos x)^2 = (1 + \sin 2x)^2$$

$$\therefore \qquad \sin^2 x + \cos^2 x + 2 \sin x \cos x = (1 + \sin 2x)^2$$

$$\therefore \qquad 1 + \sin 2x = (1 + \sin 2x)^2 = 1 + 2 \sin 2x + \sin^2 2x$$

$$\therefore \qquad \sin 2x + \sin^2 2x = 0$$

$$\therefore \qquad \sin 2x(1 + \sin 2x) = 0.$$

Hence $\sin 2x = 0$ or $\sin 2x = -1$.

$$\therefore \qquad x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{3\pi}{4}, \frac{7\pi}{4}. \qquad (1)$$

There are too many solutions - what has gone wrong? Well, by squaring, we are also solving the equation

$$\sin x + \cos x = -(1 + 2 \sin 2x),$$

and we must check back which solution fits which equation. It is easy to check that the solutions of

$$\sin x + \cos x = 1 + 2 \sin 2x \quad \text{are} \quad \boxed{0, \frac{\pi}{2}, 2\pi, \frac{3\pi}{4}, \frac{7\pi}{4},}$$

and

$$\sin x + \cos x = -(1 + 2 \sin 2x) \quad \text{are} \quad \boxed{\pi, \frac{3\pi}{2}, \frac{3\pi}{4}, \frac{7\pi}{4}}.$$

The seven solutions in (1) are the union of those two sets.

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