SOLUTIONS TO PROBLEMS FROM VOLUME 18 NO. 2

Q.527. On a sheet of paper we read the following 100 statements:-

1. On this sheet exactly one statement is false.
2. On this sheet exactly two statements are false.

. ....

. ....

100 - On this sheet exactly 100 statements are false.

Which (if any) of these statements is true?

Solution: Since the statements are mutually contradictory, the number of true statements is either 0 or 1. However, 0 statements true ⇒ 100 statements false ⇒ the last statement is true, a contradiction.

Hence the only possibility is 1 statement true; namely, the second last one asserting that 99 of the statements are false.

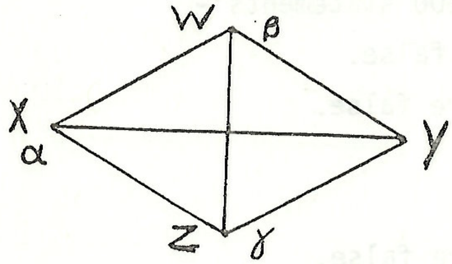
Q.528. Tom and Jim start at the same point of a race circuit and run in opposite directions round the track until Tom completes n laps, where n is an integer. Assume that both runners move at a constant speed and that Tom runs 3 laps in the time it takes Jim to complete 5. Express in terms of n the number of times the runners have met.

Solution: Note that between successive meetings of the runners the total distance covered by both together is exactly one lap. While Tom runs n laps, Jim runs $\frac{5n}{3}$ laps, and the total distance covered by both is $(n + \frac{5n}{3})$ laps $= \frac{8n}{3}$ laps. Hence they will have met $[\frac{8n}{3}]$ times, where $[x]$ denotes the largest whole number not exceedin x. (This does not count their starting position as a "meeting").

Q.529. Colour each point of the plane with one of three colours. Prove that there is a segment of length 1 whose endpoints have the same colour.

Q.529.

Solution: If it were possible to colour the plane so that the colours of the end points of unit segments were always different, then any two points X, Y whose distance apart is $\sqrt{3}$ units must have the same colour. In the diagram two equilateral triangles with unit sides, XWZ and YWZ, have been constructed. If X is coloured with colour α, the other two colours β and γ must be used for W and Z, which forces Y to be again coloured with α.

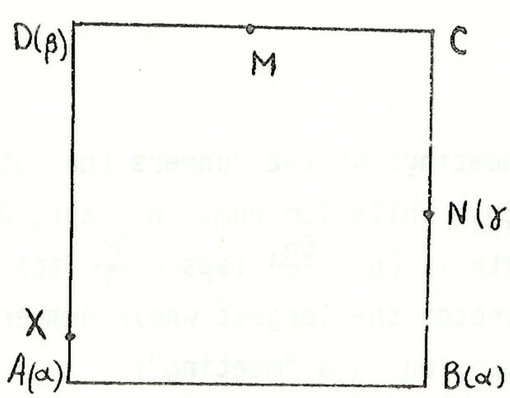


This established, the rest is easy. Consider the circle, C, centre X, radius $\sqrt{3}$. All points such as Y on its circumference must have colour α: But the circle centre Y radius 1 intersects C, at two points, P, Q. Then PY is a unit interval whose end points are both coloured with α.

Q.530. We colour each point of a unit square (including the boundary) with one of three colours. Prove that there always will be 2 points of the same colour which have a distance at least $\sqrt{65}/8$.

Solution: Of the four corners of the square at least two must have the same colour, α say. Since diagonally opposite corners are greater than $\sqrt{65}/8$ apart we may assume that exactly 2 adjacent corners, A and B in the diagram are coloured with α, and another corner, D, with a second colour, β. Suppose C is coloured with β. The mid point of BC, N, whose distance from A and from D are each greater than $\sqrt{65}/8$ can be assumed to be coloured with γ. But then the point X, $\frac{1}{8}$ th of a unit from A on the side AD has distances of $\sqrt{65}/8$ or more from each of B(α), N(γ), and C(β). We have an appropriate pair of points whichever colour is used on X.

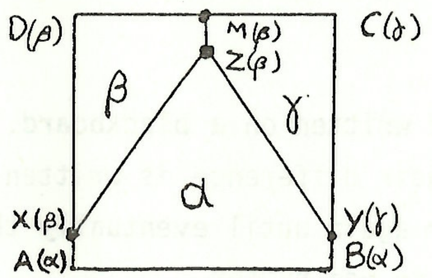
FIG. 1



So let C be coloured with γ instead of β. The mid-point M of DC whose distance from A exceeds $\sqrt{65}/8$ can be assumed coloured with β. But then X has distances of $\sqrt{65}/8$ from B(α) and M(β), and a greater distance from C(γ). Hence again we have an appropriate pair of points, whichever colour is given to X.

The distance $\sqrt{65/8}$ is the largest that can be used in the question. If the square is coloured as in Fig. 2 there is no pair of similarly coloured points whose distance exceeds $\sqrt{65/8}$. M and X are the only two points at this distance which are similarly coloured.

FIG. 2



Q.531. Five married couples meet at a party and a number of handshakes take place. No one shakes hands with his wife and no one shakes hands twice with the same person. Mr. Smith asks each of the other people present how many handshakes he (or she) has made and discovers that no two of the 9 answers are the same. How many handshakes did Mr. Smith himself make?

Solution: Clearly, since no one shakes hands with his wife, the number of handshakes made by anyone cannot exceed 8. The 9 different answers given to Mr. Smith must therefore have been 0,1,2,3,4,5,6,7 and 8. Label the couples besides the Smiths A_1, A_2; B_1, B_2; C_1, C_2; D_1, D_2. Which person shook hands 8 times? Not Mrs. Smith, since she would have shaken hands with all the "labelled" people, and then there is no-one who shook hands 0 times. So we can assume without loss of generality that it was D_2 who shook hands 8 times, i.e. with Mr & Mrs. Smith, A_1, A_2, B_1, B_2, C_1 and C_2. Then D_1 is the only person who can have made 0 handshakes.

We next ask which person shook hands with 7 people. Again not Mrs. Smith (since the seven would have had to be A_1, A_2, B_1, B_2, C_1, C_2 and D_2) because that leaves no one who shook hands only once (e.g. A_1 is already known to have shaken hands with both D_2 and Mrs. Smith). Again without loss of generality one can assume that it was C_2 who shook hands 7 times (with Mr. & Mrs. Smith, A_1, A_2, B_1, B_2 and D_2) and then C_1 is the only person who could have made just one handshake (with D_2).

Similarly we can argue that Mrs. Smith did not make 6 handshakes, and if it was B_2 who did, then B_1 was the one who shook hands twice only. Finally that A_2 and A_1 made respectively 5 handshakes and 3 handshakes.

Each of Mr. and Mrs. Smith are now known to have shaken hands with D_2, C_2, B_2 and A_2 and not to have shaken hands with D_1, C_1, B_1 and A_1.

Thus Mr. Smith shook hands 4 times.

Q.532. The numbers {1, 2, 3, 4,...., 1982} are written on a blackboard. Any 2 of the numbers are chosen. They are erased, but their difference is written on the board. This operation is repeated over and over again until eventually there remains only one number on the board. Prove that it is an odd number.

Solution: To start with there are 991 odd numbers and 991 even numbers. At each step, the number of odd numbers either stays the same (if the two numbers erased are both even, or one even and one odd) or else it decreases by 2 (if the numbers erased are both odd). Hence the number of odd numbers on the blackboard is always odd. Hence the final number on the board must be odd.

Q.533. The list of numbers 1 9 8 2 0 1 8 3 4 6 7 5 1 1 8 is construced as follows:- The first pair of digits is 19, the next pair is 82. Adding 19 and 82 gives 101, of which the last two digits of 82 + 01; and so on. Show that no four consecutive digits in this list spell out 1983.

Solution: We use E or Θ (for even and odd respectively) in place of the "tens" digit of each pair. The sequence commences

Θ9, E2, E1, E3, E4, E7, Θ1, Θ8, E9, E7, Θ6, E3, Θ9, E2, . . . It should be evident how to obtain the successive entries in this list. For example after the 8th and 9th entries Θ8, E9 the next entry has "units" digit obtained from 8 + 9, and the "tens" digit is then determined by (carried) 1 + Θ + E, an even number. Hence the 10th entry is E7 as shown

Note that the 13th and 14th entries are equal respectively to the 1st and 2nd entries. Thus this (modified) list cycles after the 12th entry. Without repeating calculations we know that the next entries will be E1, E3, ... etc.

We are now ready to ask whether the given list can contain four successive digits 1983 if it is continued indefinitely. They could occur as the pair 19, followed by the pair 83; or alternatively as a pair -1, followed by a pair 98, followed by a pair 3-.

The first possibility would imply that our modified list contained successive pairs Θ9, E3; which is not the case however far it is extended. (In fact, Θ9 is always followed by E2). The second possibility would imply that three successive pairs in the modified list were -1, Θ8, Θ-. It can be immediately seen that every occurrence of a pair Θ8 is followed by an even digit (in fact, a pair E9).

Hence neither possibility actually occurs.

Q.534. The inhabitants of a strange planet use only the letters A and O. To avoid confusion, any two words which consists of the same number of letters differ from each other in at least 3 places. Prove that the language contains at most $2n/(n + 1)$ words with n letters.

Solution: Let w represent any word of n-letters, and denote by S_w the collection of $(n + 1)$ n-letter strings which differ from w in at most one place. [For example, if w is the five letter word AOOAO then S_w comprises AOOAO, OOOAO, AAOAO, AOAAO, AOOOO, and AOOAA].

Now if u and v are any two different n-letter words, note that S_u and S_v share no letter string in common. Suppose on the contrary that "t" is a letter string common to S_u and S_v. Then u differs from the spelling of t in at most one place, whence u and v could differ in at most two places, contradicting the data.

It is now clear that amongst the 2^n strings of length n, since each word is associated with its own exclusive subset of $(n + 1)$ n-letter strings, the number of words cannot exceed $2^n/(n + 1)$.

Q.535. We want to construct letters from the Morse symbols (dots and dashes) such that (i) each letter consists of a string of 7 symbols, and (ii) if one of the seven symbols in a letter was transmitted erroneously, the letter should still be recognizable. (i.e. the string is more like the correct string than any of the other strings used for letters.) How many letters can this alphabet contain?

Solution: In order that a single transmission error still leaves identifiable the correct letter, it is clear that two different letters must be represented by strings differing in at least three places. The previous question (with "word" replaced by "letter", and "letters A and O" replaced by "symbols dash and dot") then shows that the number of different 7-symbol letters is at most $2^7/(7+1) = 16$.

Of course, this doesn't prove yet that 16 such 7-symbol strings can actually be found. However it is not particularly difficult to construct examples, such as the following; (a dot is represented by 0, a dash by 1):-

0 0 0 0 0 0 0	1 1 1 1 1 1 1
1 1 1 0 0 0 0	0 0 0 1 1 1 1
1 0 0 1 1 0 0	0 1 1 0 0 1 1
1 0 0 0 0 1 1	0 1 1 1 1 0 0
0 1 0 1 0 0 1	1 0 1 0 1 1 0
0 1 0 0 1 1 0	1 0 1 1 0 0 1
0 0 1 0 1 0 1	1 1 0 1 0 1 0
0 0 1 1 0 1 0	1 1 0 0 1 0 1

Q.536. A number of circles are drawn in a plane in such a manner that there are exactly 12 points in the plane that lie on more than one circle. What is the smallest number of areas into which the plane can be subdivided by the circles?

Solution: We shall prove more generally that if there are n-points lying on more than one circle, the minimum number of areas is $(n + 2)$. It is sufficient to prove this on the assumption that the circles are not in two or more disconnected "clumps" as in Fig. 1., for if the two parts of the diagram are moved together until two of the circles touch (point P now coinciding with Q) the number of "two circle points" has increased, but the number of regions has not changed.

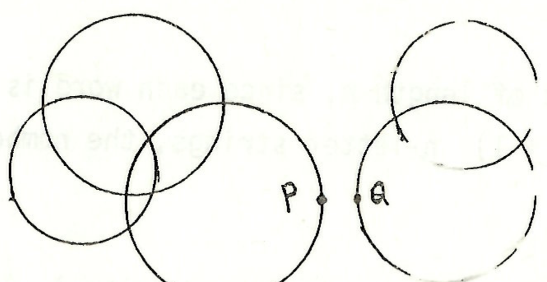


FIG. 1

For such "connected maps" in the plane, the number of regions, F, is related to the number of vertices, V (points lying on two or more circles), and the number of edges, E, (arcs of circles terminated at each end by a vertex) by the well-known formula of Euler: $F - E + V = 2$. (A proof of this formula may be found in "What is Mathematics" by Courant and Robbins).

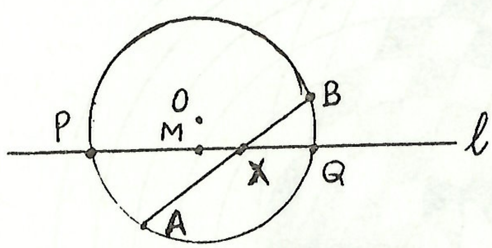
Since each vertex lies on at least two circles, there are at least 4 "ends" of edges at that vertex. As each edge has two ends, and there are n vertices, the number of edges must be at least $\frac{1}{2} \times 4n$.

Hence $F(= E - V + 2)$ is at least $2n - n + 2 = n + 2$. (In fact, our argument shows that for every such connected map, the number of regions is exactly $n + 2$ provided that no point lies on more than two circles.).

Taking $n = 12$, the smallest number of areas possible is 14.

Q.537. A and B are points lying on opposite sides of a given line ℓ. How can one construct the circle passing through A and B having the shortest possible chord on the line ℓ ?

Solution: In the diagram, P and Q are points of intersection with ℓ of a circle through A and B, and M is the mid point of PQ. It is a well known theorem of geometry that for the two chords intersecting at X, $PX.XQ = AX.XB$.



i.e. $(PM + MX).(PM - MX) = AX.XB$

$\therefore \quad PM^2 = AX.XB + MX^2$.

Since $AX.XB$ is invariable, the smallest value of PM is obtained when $MX = 0$; i.e. when X is the mid point of the chord PQ. The centre, O, of the circle can be readily constructed as the point of intersection of the perpendicular bisector of AB with the line perpendicular to ℓ at the point X.

Q.538. If a,b are given non-zero numbers, find all values of x such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{x} = \frac{1}{a+b+x} .$$

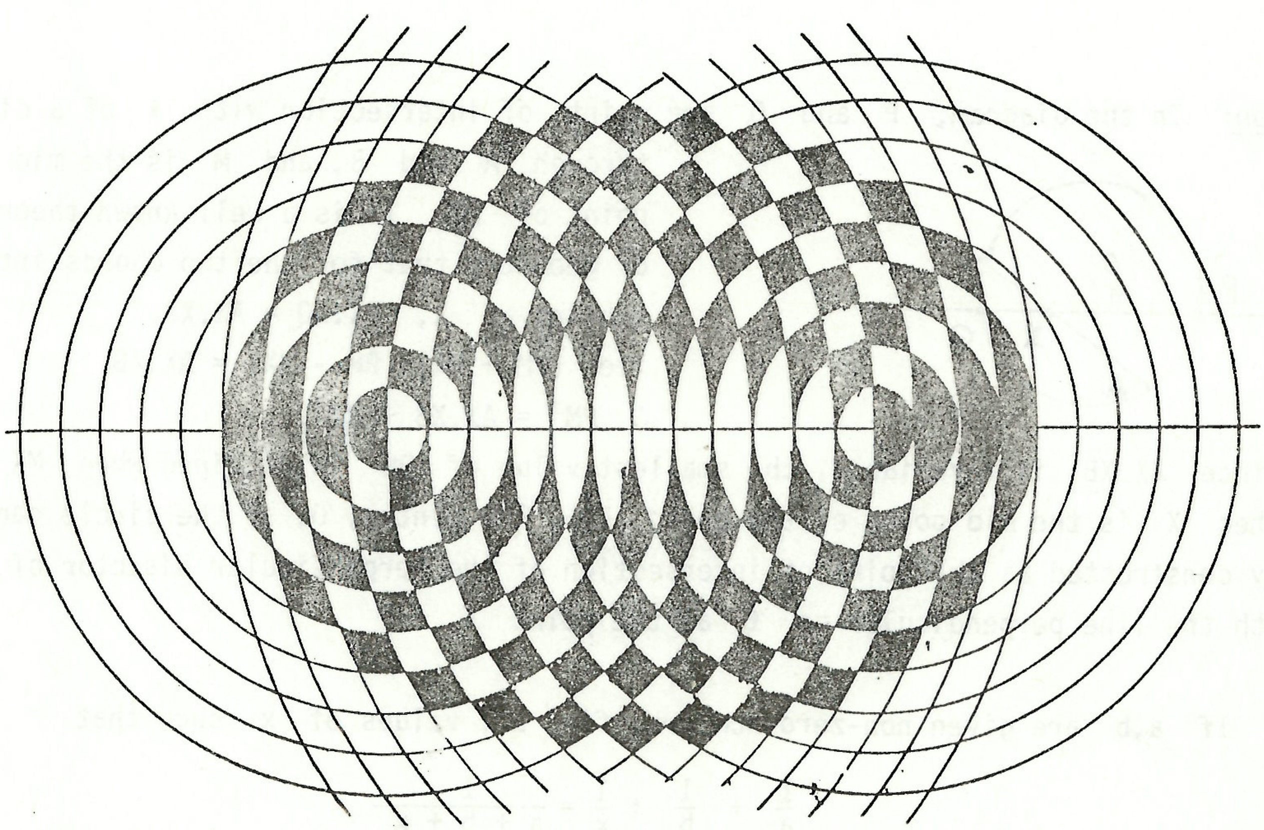
Solution: $\frac{1}{a} + \frac{1}{b} = \frac{1}{a + b + x} - \frac{1}{x}$

$$\Leftrightarrow \frac{a+b}{ab} = \frac{-(a+b)}{x(a+b+x)}$$

$$\Leftrightarrow a + b = 0 \quad \text{or} \quad ab = -x(a + b + x)$$

If $a + b = 0$, the equation is satisfied by every non-zero value of x. Otherwise x is a root of $x^2 + (a + b)x + ab = 0$; i.e. $x = -a$ or $x = -b$.

PROBLEM SOLVERS: R. Bozier (Barker College) 527, 528, 532, 534, 535*, 538*. S. Evans (St. Ignatuis College, Riverview) 527, 538*. Janine Mok (P.L.C.) 532. (A * indicates that the solution supplied was judged to be incomplete.)



The two families of concentric circles form at their intersection a family of ellipsies. Can you see a family of another circle?