THE GOLDEN RATIO REVISITED

A while ago we received a short article from one of our readers, Tim Bedding about some features of this celebrated ratio. We are thankful for his reminder and decided to incorporate his work into a somewhat wider - by no means exhaustive - review of the Golden Ratio and its ramifications.

The quantitative comparison of two magnitudes constitutes a ratio, e.g. $\frac{a}{b}$, while a proportion is the equality of ratios between two pairs of quantities, e.g. $\frac{a}{b} = \frac{c}{d}$. Proportions where the quantities a,b,c,d are linked with each other by some further relationships were of great interest to geometers, artists and architects throughout the ages. Pythagoras, who is probably better known to you for his theorem concerning right angled triangles, also discovered that the musical consonances could be produced by dividing the string of a lyra (the Greek's favourite musical instrument) into invariably fixed ratios. The discovery of such a close connection between "numbers" and "harmony" or "beauty" led to the formidable edifice of the Pythagorian-Platonic school of thought, and as they never fought shy of applying "mathematical" notions to "non-mathematical" phenomena they were convinced they had found the key to the mysteries of the macrocosm and microcosm. Their "mathematical" underpinning of philosophy and cosmology exerted the greatest influence on western thought.

The proportions (or ratios) connected with musical instruments can be expressed by integers or simple fractions, but many of the geometrical proportions lead to the "incommensurable" or "irrational". For example: you are familiar with the ratio of a side to the hypotenuse in an isosceles right angled triangle:

$$\frac{\text{side}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}}$$

The construction of a regular pentagon depends on the "cut in extreme and mean ratio", i.e. in the proportion of the Golden Section as it was discussed by Euclid in his "Elements", the granddaddy of all bestsellers. This proportion contains only two magnitudes (hence the varied nomenclatura of ratio / proportion) and is expressed as

$$\frac{a + b}{a} = \frac{a}{b}$$

It is usual to attribute the explicit appearance of the Golden Section to Euclid, but we have evidence that it was known to the Egyptians before them, as it is exemplified by Fig.1.

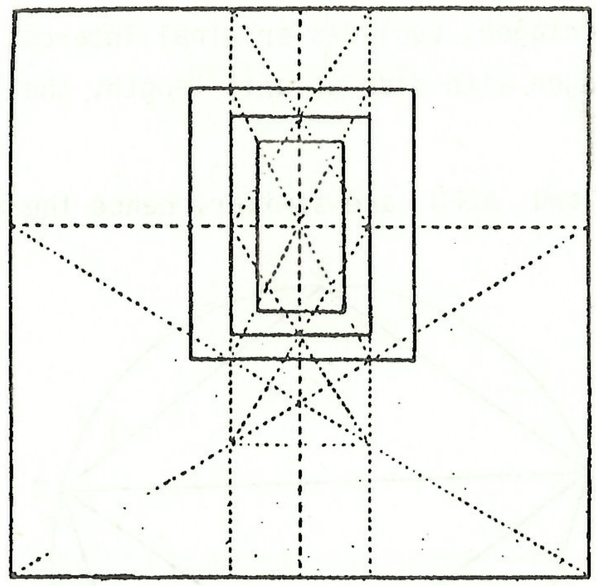
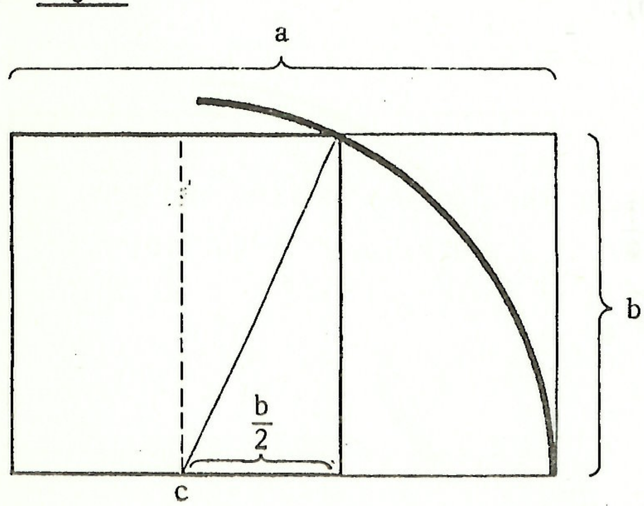


Fig.1. Tomb of Rameses IV. The diagram shows the analysis of the tomb and its triple sarcophagus after a contemporary papyrus. The innermost sarcophagus is a double square, the middle one is a rectangle with sides conforming to the Golden Ratio, the outer one is composed of two such Golden Rectangles each equal to the middle one.

We may construct the Golden Ratio by constructing the Golden Rectangle as shown in Fig.2.



$$\frac{a}{b} = \phi$$

Fig. 2.

Its numerical value may be calculated from the quadratic equation arising from

$\frac{a+b}{a} = \frac{a}{b}$, i.e. $\left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right) - 1 = 0$ with roots $\frac{1 \pm \sqrt{5}}{2}$, and omitting the negative root we get the Golden Number

$$\phi = \frac{1 + \sqrt{5}}{2} = 1.618033989\text{---}$$

Returning to the regular pentagon, Euclid's original interest, it is not difficult to see that in a regular pentagon with side of unit length, the length of a diagonal is ϕ.

In Fig.3. triangles ΔAFC and ΔEFD are similar, hence the ratios of corresponding sides are equal.

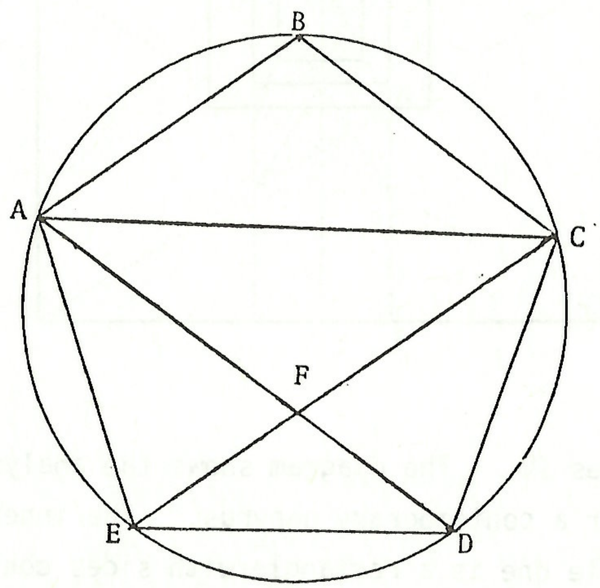


Fig.3.

Furthermore AC = AD (equal diagonals)
ED = BC (sides of regular pentagon)
BC = AF (sides of parallelogram).

These data yield the string of ratios.

$$\frac{AF}{FD} = \frac{AC}{ED} = \frac{AD}{BC} = \frac{AD}{AF} = \phi$$

$$AF = BC = 1, \quad AD = \phi, \quad FD = \frac{1}{\phi}$$

$$\frac{AF + FD}{AF} = \phi \quad \text{and} \quad \frac{1 + \frac{1}{\phi}}{1} = \phi$$

yielding the equation

$\phi^2 - \phi - 1 = 0$, its positive root the Golden Number.

The occurrence of the Golden Number in the regular pentagon elevated this figure and its derivatives: the pentagram, the decagon and the star-decagon into the realm of cosmic and religious mystery, as exemplified by Fig.4 and Fig.5.

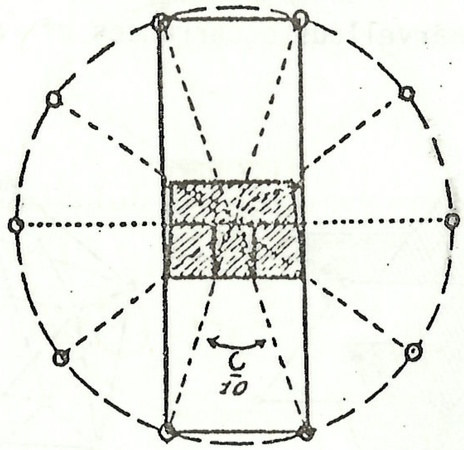


Fig. 4 The plan of an Egyptian Temple

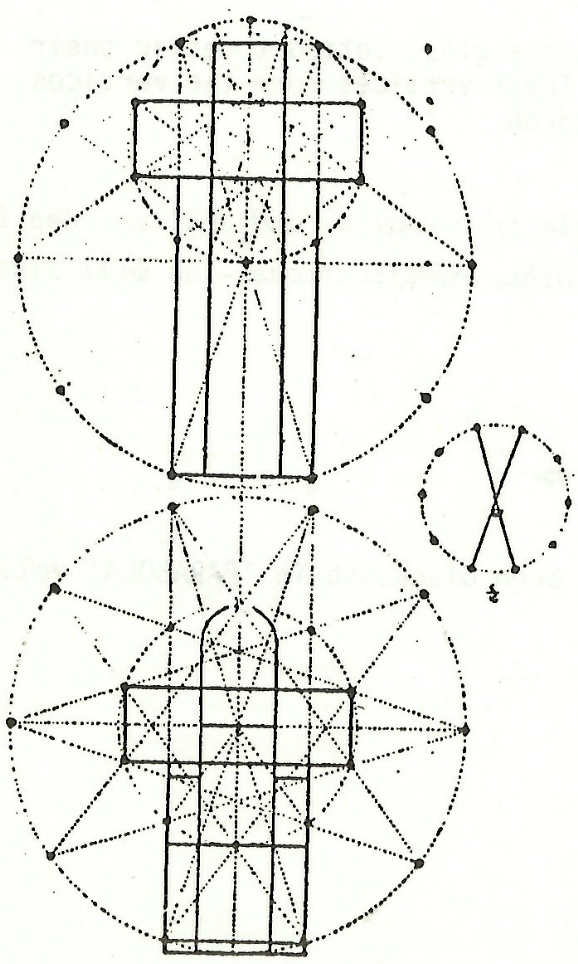


Fig. 5. Diagrams showing two standard plans for gothic cathedrals (from Moessel).

These plans indicate a "circle of orientation", probably traced on the ground itself, the circumference divided into 5, 10 or 20 equal parts clearly yields the desired proportion and related proportions of the Golden Number.

The aesthetic-religious influence of this Golden Ratio flourished during the Renaissance, indeed a beautiful book wirtten by Luca Pacioli (1445 - 1509) under the influence of Piero della Francesca (ca. 1416 - 1492) and illustrated by Leonardo da Vinci collected many of the marvellous occurrences of ϕ, one of them illustrated in Fig. 6.

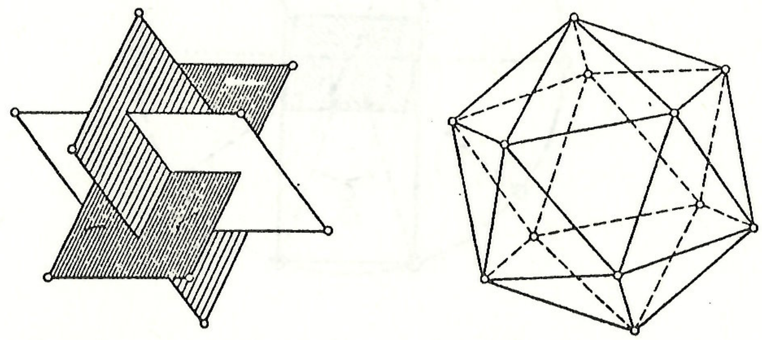


Fig. 6. The Golden Rectangles slotted together their centres coinciding. Their vertices form the vertices of the regular icosahedron.

Successive subdivision of the Golden Rectangle into smaller and smaller ones indicate the existence of a spiral curve which - with other ramifications - we will discuss in our next issue.

Spirals connected with the Golden Ratio has been discussed in 'PARABOLA' volume 16 No. 3, from a different view point.